Frictional Capital Markets and Economic Growth*

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Abstract

This paper develops an endogenous growth model where firms face idiosyncratic productivity shocks, necessitating capital reallocation. Capital goods trade in a frictional dealer market, giving rise to a distribution of capital holdings and Tobin's q across firms. The model highlights the intricate relationship between capital misallocation and growth. Absent externalities, market frictions worsen capital misallocation and hinder growth. When brokers are perfect monopolists, vanishing search frictions eliminate capital misallocation but leave growth inefficiently low. With learning-by-doing externalities, reducing frictions lowers misallocation but has nonmonotonic effects on growth and welfare. With entry, higher dealer's bargaining power can reduce misallocation and growth.

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1 Introduction

The accumulation and allocation of capital are fundamental in explaining output levels and growth rates. In textbook models, capital goods are efficiently traded in Walrasian markets and optimally distributed across firms to maximize output. In practice, however, capital is continually reallocated from downsizing or exiting firms to those entering or expanding, and this process, which accounts for one-third of total capital expenditures (Cui, 2022), is far from seamless (e.g., Ramey and Shapiro, 2001). Indeed, the secondary markets for capital goods, where capital reallocation takes place, share many of the same frictions as those that plague over-the-counter (OTC) financial asset markets: they are decentralized, fragmented, and subject to significant delays.[1] On Surplus Record, a leading directory for used machinery and industrial equipment, the turnover rate for the most popular machines (the ratio of removed listings to total listings) ranges from 5% to 10% per month. Similarly, on MachineryTrader, a platform for buying and selling construction equipment, monthly turnover rates range from 5% to 30%. [2] Moreover, the reallocation process is facilitated by dealers and brokers, who typically possess significant market power to determine prices.[3]

The objective of this paper is to develop a parsimonious and tractable model of endogenous growth with firm heterogeneity that rigorously incorporates the microstructure of the capital goods market and its frictional nature. The paper explores the intricate relationship between capital reallocation and economic growth by addressing the following questions: How do misallocation and growth correlate across economies with different capital market structures? Do market frictions that exacerbate capital misallocation necessarily hinder long-term growth? If financial markets are efficient, how do frictions in the market for physical capital affect the rate of return on financial claims relative to capital productivity? Finally, what would be the welfare gains from eliminating frictions in secondary capital markets?

The benchmark model depicts an economy with firms operating technologies that feature a constant marginal product of capital. They differ in their idiosyncratic productivities, which evolve stochastically over time. Households own financial claims (e.g., equity shares),

[1] The importance of search frictions in the market for used capital is discussed, for instance, by Ramey and Shapiro (2001) in the context of the aerospace industry. The role of secondary markets for capital goods has also been highlighted by the supply chain disruptions of the 2020-21 pandemic (Darmouni and Sutherland, 2024).

[2] I thank Tom Scanlan, the director at Surplus Record, and Eric Korth, from MachineryTrader, for providing me with this data.

[3] Gavazza (2016) mentions the presence of independent brokers to match buyers and sellers in the market for commercial aircrafts and argues that "intermediaries play an important role in mediating transactions. Most intermediaries operate as brokers who match buyers and sellers." The magazine Surplus Record in its edition of November 2024 (pages 7 to 14) provides a list of machinery and equipment dealers/brokers for different states in the US.

while firms own and manage the capital they operate with the goal of maximizing their market value. Capital is traded in a frictional dealer market, as formalized in the literature on OTC markets (e.g., Duffie, Gârleanu, and Pedersen, 2005; Lagos and Rocheteau, 2009). In this setup, firms negotiate with brokers the terms at which they buy or sell capital. Brokers form a frictionless network and trade competitively among themselves.

The model admits a unique balanced-growth path equilibrium, which features two forms of misallocation: static and dynamic. *Static misallocation* occurs when aggregate output, or total factor productivity (TFP), falls below its first-best level ($Y < Y^*$), given the capital stock. It is due primarily to search frictions that cause some capital to remain with low-productivity firms in equilibrium even though it is optimal for all capital to be operated by high-productivity firms. The extent of this misallocation depends on the search technology, the frequency of idiosyncratic productivity shocks, and the endogenous capital growth rate. Higher growth reduces misallocation because newly produced capital is entirely allocated to the most productive firms, whereas some of the old capital remains with less productive firms.

Dynamic misallocation arises when the output growth rate is below its first-best level ($g < g^*$). It occurs when the equilibrium real interest rate is lower than the marginal product of capital at the most productive firms. The gap between the real interest rate and its frictionless value, referred to as the *illiquidity discount*, stems from search frictions and the inability to instantly reallocate capital when firms experience negative productivity shocks—the same frictions generating static misallocation. In addition, the illiquidity discount increases with the bargaining power of dealers due to an intertemporal holdup problem. I show that when search frictions are small but dealer's bargaining power is high, the rate of return of financial claims is less than the average productivity of capital, which can help explain why some countries experience low growth rates despite a high productivity of their capital.

Misallocation is defined relative to a first-best outcome without market frictions. However, because search frictions are features of the environment that are largely inescapable for the policymaker, I also compare the equilibrium allocation to the solution of a social planner's problem under the same technological constraints faced by brokers. I show that the constrained-efficient allocation coincides with the decentralized equilibrium when brokers possess zero bargaining power.

Finally, I examine how misallocation evolves as search frictions disappear. This limiting case can be interpreted as the market shifting to online platforms and becoming less segmented. If brokers lack full bargaining power, the equilibrium converges to the Walrasian equilibrium of the textbook AK model, where "A" represents the productivity of the most efficient firms. Thus, as search frictions vanish, both static and dynamic misallocation disappear. However, when brokers are perfect monopolists with complete information and full bargaining power, the real interest rate remains inefficiently low, below the productivity of capital. In that case, static misallocation vanishes, but dynamic misallocation persists.

In the second part of the paper, given its focus on the market for capital goods and growth, I link knowledge creation to capital accumulation, following Arrow (1962), Frankel (1962), and Romer (1986). As firms install new capital, they gain experience in operating it and discover ways to produce more efficiently. This experience, represented by cumulated investment, is nonrival and generates spillovers across firms. At the frictionless competitive equilibrium, although capital is allocated efficiently across firms, the rate of capital accumulation remains inefficiently low.

When the capital market is frictional, firms encounter a holdup problem because they must frequently adjust their capital, through brokers or dealers, due to idiosyncratic shocks and aggregate productivity growth. To mitigate exposure to this holdup problem, firms adopt an optimal strategy that worsens static misallocation but alleviates dynamic misallocation. To see why it worsens static misallocation, note that in order to minimize their need to trade amid idiosyncratic shocks, firms hedge by selecting a capital level aligned with their average productivity over time. High-productivity firms reduce their desired capital holdings as frictions rise, whereas low-productivity firms increase theirs. Consequently, in equilibrium, an excessively large share of the capital stock is allocated to the least productive firms, whereas the most productive firms receive an insufficient share of the overall capital stock.

At the same time, to counteract the holdup problem from rising aggregate productivity, firms frontload capital acquisitions. This strategy partially offsets the learning-by-doing externality and can reduce dynamic misallocation. As a result of these opposing forces—the hedging and the frontloading—the real interest rate and the economy's growth rate can vary non-monotonically with search frictions and the bargaining power of brokers. In certain cases—such as when brokers have high bargaining power and idiosyncratic productivity risk is low—greater market frictions can boost economic growth, even exceeding the Walrasian equilibrium. Since market frictions impact static and dynamic misallocations in opposite ways, their overall effect on welfare can be positive or negative, e.g., if the idiosyncratic productivity risk is small, high broker's bargaining power raises welfare. In a calibrated version of the model, removing all search frictions worsens society's welfare.

In the final part of the paper, I provide yet another example of the complex relationship between misallocation and growth. I endogenize the capital market structure by providing microfoundations for search frictions. While more bargaining power to dealers can speed up reallocation and reduce misallocation, it may also lower the rate of return of financial wealth and the economy's growth rate.

1.1 A brief history of the secondary market for capital goods

The production of capital goods, such as machinery and heavy equipment, was pivotal to the growth of the U.S. economy during the 19th and 20th centuries (Rosenberg, 1963). The rapid evolution of machines gave rise to a market for used machinery. Entrepreneurs, inspired by the junkyard industry, sought out discarded or damaged machines, repaired and repurposed them, and sold them for profit.[4] According to Graff (2002),

> "As the United States industrialized in the 1920's and 1930's, a need grew for some kind of distribution of cheap machine tools for companies that needed equipment quickly, cheaply, or both. Enter the entrepreneurs who saw opportunity in the marketplace. They bought up the discards of the bigger factories."

As an anecdote illustrating the growth of the secondary market in the 1920s, the magazine Surplus Record was created in November 1924, nearly 100 years ago. The objective of its founder was to provide a publication cataloging surplus equipment to connect buyers and sellers. It allowed dealers to list their surplus equipment and provided buyers with a convenient resource for finding used capital goods. It was the ancestor of online marketplaces.

World War II was an inflection point for the development of the market for used equipment. The Machinery Dealers National Association (MDNA) was established at the beginning of the war, in 1941, to provide a network of used machinery dealers. Today, the association represents more than 400 members throughout the world. At the end of the war, over $100 billion of wartime machinery (trucks, aircraft engines, machine tools, generators, etc.) needed repurposing for civilian manufacturing.[5] The Machinery Dealer National Association lobbied the U.S. Congress to play a role in the redeployment of this equipment. This effort was successful, and in December 1945, the Office of Reconstruction Finance Corporation announced a 12.5% sales commission for any used machinery salesman who sold a surplus war machine. From Graff (2002):

> "Many young men got into the [machinery] business after the war selling off U.S. Government-owned machine tools on a commission. It required very little capital and the Government had mountains of equipment to sell."

[4] On the evolution of junkyards, see https://www.junkyardslist.com/junkyard-basics/evolution/.

[5] See https://surplusrecord.com/about/history/

From the 1960s onward, dealers became more specialized and the market benefited somewhat from the globalization of trade, as developing countries became important buyers of used machinery from developed nations. The advent of the internet in the late 1990s triggered another significant transformation of the secondary market for machines. Prospective buyers of machinery and equipment once had to visit dealerships in person to find and inspect the machines they sought. Major cities—e.g., Chicago, New York, and Los Angeles—had streets dedicated to machinery dealers. The internet has enabled the rise of online marketplaces, such as MachineryTrader.com (launched in 1998) and IronPlanet.com (founded in 1999), among others, where buyers and sellers can meet virtually, significantly expanding the market. Moreover, auctions of machines and equipment that once took place in person at physical locations can now be conducted online.[6] In addition, intermediaries no longer need to hold inventories, as they can act as brokers who connect buyers and sellers.

The development of secondary markets is not limited to tangible capital. One example of this is the secondary market for patents. According to Kim and Valentine (2022), "29% or 1.3 million patents of the 4.5 million patents that the U.S. Patent and Trademark Office granted from 1994 to 2017 were sold to another party at least once by 2020." The patent market is described as decentralized and informationally opaque. According to Mossoff (2015), the secondary patent market emerged in the 19th century to allow innovators to trade their ideas. Intermediaries called *patent agents* were active in that market since the late 19th century. Nowadays, the secondary patent market is organized by brokers. According to Richardson, Costa, and Oliver (2024):

> "Brokers are critical to the functioning of this market. We cannot emphasize this enough. (...) They sell, buy, market, negotiate, evangelise, and promote patents in the market."

1.2 Related literature

The literature on capital reallocation under adjustment costs, e.g., Eisfeldt and Rampini (2006), shares similarities with the approach in this paper.[7] The closest paper in that literature is Eberly and Wang (2009) who study reallocation and growth in a two-sector AK

[6] For the evolution of industrial auctions, see https://hub.exapro.com/the-evolution-of-industrial-auctions-from-physical-to-online-platforms-since-the-1990s

[7] The literature on capital reallocation and misallocation is not limited to adjustment costs. Existing models explain misallocation with different features, including adverse selection (e.g., Eistfeld, 2004), agency costs (e.g., Eistfeld and Rampini, 2008), and search frictions. David and Venkateswaran (2019) provide a methodology to disentangle the different sources of capital misallocation within a general equilibrium model of firm dynamics. An overview of the stylized facts and the different theoretical approaches is provided by Eisfeldt and Shi (2018).

model.[8] While my model can be viewed as providing microfoundations for these adjustment costs by diving into the micro-structure of the capital market, it goes beyond rationalizing these costs. It separates costs that are technological in nature, e.g., the infrequent matching of firms and brokers, from the costs arising from market power, and it shows they affect TFP and the rate of return of financial claims differently. For instance, dealers' market power does not directly affect TFP; rather, it alters the rate of return on financial wealth through an intertemporal holdup problem, which has significant implications for growth when combined with knowledge externalities.[9]

The description of the market structure follows the work of Duffie, Gârleanu, and Pedersen (2005, 2007) for over-the-counter (OTC) markets. I adopt the version with divisible and unrestricted asset holdings of Lagos and Rocheteau (2007, 2009). A thorough and in-detail treatment of this class of models is proposed by Hugonnier, Lester, and Weill (2025). The relevant chapters are Chapter 4 on semi-centralized markets and Chapter 6 on divisible asset holdings. Compared to the literature on OTC markets, this paper embeds the OTC market structure within a general equilibrium framework. This permits to endogenize the rate of return of financial claims, which is used to compute the financial value of firms and the distribution of Tobin's q. Moreover, it endogenizes the asset supply and the growth rate of capital. Finally, while utility flows from holding assets are stationary and presented in reduced form in most of the literature, they depend on technology in my model and grow with aggregate productivity at an endogenous rate.

The OTC market structure has been used to describe real asset markets. For instance, Gavazza (2011, 2016) applies a version of Duffie, Gârleanu, and Pedersen (2005, 2007) to the market for commercial aircrafts. Nosal and Rocheteau (2011, Chapter 12), Kurmann (2014), and Kurmann and Rabonovich (2018) use a market structure similar to the one in Lagos and Rocheteau (2009) to describe the market for physical capital. Ottonello (2024) adopts a competitive-search version of the Mortensen and Pissarides (1994) model to study capital unemployment over the business cycles.[10] Bierdel, Drenik, Herreño, and Ottonello (2023) extend the model to add informational asymmetries about the quality of capital goods.

[8] Dou, Ji, Tian, and Wang (2024) study misallocation and (Schumpeterian) growth in a model with heterogeneous firms facing financial frictions due to agency problems. Jones (2013) discusses the link between misallocation and income differences across countries and emphasizes the potential role of intermediate goods and input-output linkages. Jovanovic (2014) studies growth via on-the-job learning and misallocation of heterogeneous workers in an overlapping generations model in which young agents match with the old.

[9] Frictions in the capital market constitute external adjustment costs. For the distinction between internal and external adjustment costs see, e.g., Mussa (1977).

[10] Lagos (2006) also proposes a search-theoretic model of endogenous TFP in the spirit of Mortensen and Pissarides (1994). Lester, Rocheteau, and Weill (2015) study the models of OTC markets with dealers of Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2009) under competitive search.

The description of the market structure emphasizes the role of brokers/dealers in the reallocation of capital goods. Relatedly, Greenwood and Jovanovic (1990) formalizes the contribution of *financial* intermediaries to economic growth in an AK model where intermediaries allow households to obtain a higher and safer return. Mehra, Piguillem, and Prescott (2011) formalize costly financial intermediation in a neoclassical growth model. Intermediaries in my model do not connect households and firms but firms with different productivities. Moreover, financial markets are efficient to focus in the frictions in the capital market.

This paper introduces market power in the capital market and shows it affects the real interest rate and the growth rate of the economy. There is a vast macroeconomic literature on market power in goods markets. For instance, Ball and Mankiw (2023) study the role of market power in a neoclassical growth models and show that firms' market power in the goods market reduces the real interest rate. Relatedly, there is a literature studying misallocation in models where the market power of heterogeneous firms arises from monopolistic competition (e.g., Edmond, Midrigan, and Xu, 2023). In contrast to the monopolistic competition approach, the market power of dealers arises from search frictions and endogenous outside options and it does not generate static distortions because the outcomes of the negotiations are pairwise Pareto-efficient. Inefficiencies occur through intertemporal holdup problems that affect the valuations of firms.

In the main model, I assume that firms face no financial or liquidity constraints.[11] Wright, Xiao, and Zhu (2018, 2020) study capital reallocation in a New Monetarist model with search frictions and liquidity considerations. Cui, Wright, and Zhu (2025) extend the model to distinguish between full or partial sales and calibrate it to match business cycle facts. Silveira and Wright (2010) adopt a similar description for the market of ideas (intangible capital). Chiu, Meh, and Wright (2017) study a version of the market with ideas under financial frictions and endogenous growth.

2 Environment

Time is continuous and indexed by $t \in \mathbb{R}_+$. There is a single consumption good and a unit measure of identical, infinitely-lived households whose preferences are represented by the following lifetime utility function,

$$\int_0^{+\infty} e^{-\rho t} u(c_t;\theta) dt, \tag{1}$$

[11] See the supplementary appendix for a version where firms hold liquid assets to finance capital purchases.

where $\rho > 0$ is the rate of time preference, and c_t is the time-path for consumption. The instantaneous utility function is of the CRRA form,

$$\begin{aligned} u(c;\theta) &= \frac{c^{1-\theta}}{1-\theta} \text{ if } \theta \neq 1 \\ &= \ln(c) \text{ if } \theta = 1. \end{aligned} \tag{2}$$

Since the focus is on economic growth, I omit the choice between consumption and leisure.

Households have the possibility to acquire financial claims issued by firms, such as equity shares or bonds. Financial markets are perfectly efficient: securities are traded without any transaction costs under complete information. There is no aggregate uncertainty. The idiosyncratic risk associated with financial claims can be diversified away, e.g., through the use of mutual funds. The endogenous real rate of return on financial wealth is denoted r_t.

There are three category of firms/technologies. There is a unit measure of *capital producers* that have the technology to transform the perishable consumption good into a durable capital good, one for one. Capital does not depreciate. The aggregate capital stock is denoted $K_t \in \mathbb{R}_+$ and the production of capital goods is denoted $\dot{K}_t$, where a dot over a variable denotes a time derivative. There is no rental market for physical capital, i.e., firms own the capital they operate.[12] Capital goods are traded in a frictional broker market, the micro-structure of which is detailed below.

There is a unit measure of final good producers, simply labeled *firms.* Each firm is endowed with a technology represented by the production function, $f_j(k)$, where $k \in \mathbb{R}_+$ is the capital held by the firm, and $j \in \{L, H\}$ indicates the idiosyncratic and time-varying productivity of the firm.[13] I assume for now, that the technology is linear,

$$f_j(k) = A_j k \text{ with } 0 \leq A_L < A_H. \tag{3}$$

Later, in Section 4, I introduce diminishing private returns to capital and I consider a richer set of productivity realizations. (See also the Supplementary Appendix for a general Markov chain for A_j.) The idiosyncratic productivity shocks are formalized as follows. At Poisson arrival rate $\lambda > 0$, a firm draws a new productivity, A_j with $j \in \{L, H\}$, with probability $\pi_j > 0$ and $\pi_L + \pi_H = 1$. The average productivity is denoted by $\bar{A} = \pi_L A_L + \pi_H A_H$. The objective of each firm is to maximize its market value.

[12] One interpretation is that rental companies cannot bypass or eliminate the search frictions and therefore, for the sake of being parsiminious, they are omitted. Alternatively, one could interpret the firms in the model as rental companies and the idiosyncratic productivity shocks are the extent to which their inventories of capital goods are utilized.

[13] If productivity shocks are i.i.d., there is no loss in generality in assuming two productivity levels. If $A_L = A_H$, our model is a standard AK model. For a textbook treatment, see, e.g., Barro and Sala-i-Martin (2004).

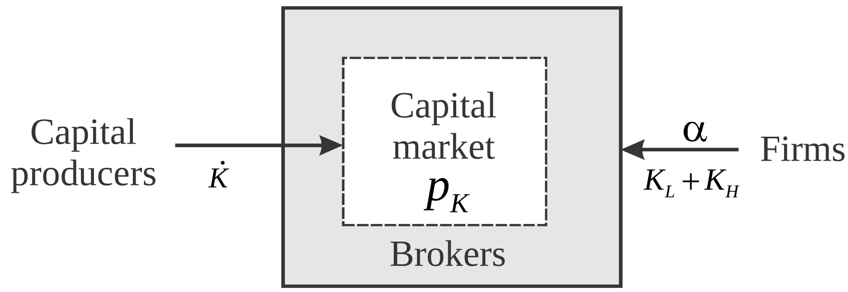

Figure 1: Description of the capital market structure

Finally, there is a unit measure of brokerage firms, simply called *brokers*, who provide intermediation services in the capital market. The market structure is represented graphically in Figure 1. Brokers, who can contact each other instantly, trade capital goods competitively among themselves.[14] The competitive price of capital among brokers, expressed in terms of the consumption good, is p_K. Firms are matched bilaterally with brokers at Poisson arrival rate $\alpha > 0$. Brokers can execute trades on behalf of firms instantly. The terms of trade between brokers and firms are negotiated.

Capital producers can trade competitively with brokers and sell the newly produced capital goods instantly at price p_K. Therefore, the new capital is allocated to firms instantly. This assumption allows us to focus on frictions in the second-hand capital market by capturing the idea that the new capital is easier to trade, e.g., because its quality is more easily established and capital producers are better connected to brokers. I relax this assumption in the Supplementary Appendix by introducing delays to sell the new capital to *primary* dealers with bargaining power in pairwise meetings.

Interpretation of the market microstructure.

This semi-centralized, over-the-counter market structure captures the observation that dealers of capital goods can contact each other quickly to match their clients' supplies and demands.[15] For instance, according to the Energy Business Review (2024), machinery brokers "*have an extensive network of contacts, including manufacturers, sellers, and other*

[14] Farboodi, Jarosch, and Shimer (2023) construct a model of an OTC market in which traders invest in a contact technology and show that middlemen with unboundedly high contact rates emerge endogenously.

[15] While I assumed that all trades go through brokers, one can also allow for business-to-business trades. Indeed, an equivalent reformulation consists in assuming that firms access the competitive market directly at rate α_d and indirectly through brokers with bargaining power η_b at rate α_b. This market structure can be mapped into the one described above when $\alpha = \alpha_m + \alpha_b$ and $1 - \eta = [\alpha_m + \alpha_b(1 - \eta_b)] / (\alpha_m + \alpha_b)$. For models of OTC markets where trades are fully decentralized and intermediation emerges endogenously, see, e.g., Üslü (2019) and Hugonnier, Lester, and Weill (2020).

brokers". Similarly, the Machinery Dealer National Association advertise that "*MDNA dealers can communicate with each other through various means*" thereby providing machinery availability.[16] The description is not limited to the market for tangible capital. In the secondary patent market, according to Richardson, Costa, and Oliver (2024), "*brokers leverage their networks of connections to find interesting patents to sell and to find willing buyers.*"

From a methodology standpoint, the competitive (interdealer) market is a formalization device to bypass the inventory problem of dealers in order to focus on their market power when trading with firms, which is captured by η, and their role in providing immediacy, which is captured by α. As we will see, this formalization keeps the model analytically tractable even when there is rich heterogeneity across firms and holdings of capital goods are perfectly divisible.

Regarding the use of bargaining to determine terms of trade between brokers and firms, the Energy Business Review (2024) mentions, among the reasons to engage industrial machinery brokers, the fact that "*brokers are expert negotiators.*" In the secondary patent market, brokers "*sell, buy, market, negotiate, evangelise, and promote patents in the market*" (Richardson, Costa, and Oliver, 2024). To the extent that one focuses on trading mechanisms under complete information, the use of generalized Nash bargaining is with no loss in generality.[17]

3 The benchmark model

The benchmark model incorporates linear technologies, no externalities, and a fixed measure of brokers. It provides a tractable setting to develop intuition and assess the impact of market frictions on allocations and growth. In the following, I characterize balanced-growth equilibria where aggregate output and capital stock grow at the same endogenous constant rate, $g > 0$.

3.1 Households

The representative household maximizes its lifetime utility given some initial financial wealth, ω_0, and a time-path for the real interest rate, r_t, i.e.,

$$\max_{c_t,\omega_t} \int_0^{+\infty} e^{-\rho t} u(c_t) dt \text{ s.t. } \dot{\omega}_t = r_t \omega_t - c_t \text{ and } \omega_0 \text{ given.} \tag{4}$$

[16] https://www.mdna.org/

[17] Indeed, the Nash solution generates the entire Pareto-frontier in pairwise meetings by varying η. For instance, it is payoff-equivalent to request-for-quote mechanisms, as shown by Maciocco (2025).

It chooses its consumption, c_t, and its investment in financial wealth, $\dot{\omega}_t$, where the portfolio of financial assets is perfectly diversified, subject to the budget identity in (4) and taking as given the rate of return of financial wealth, r_t. By the Maximum Principle, the optimal choice for consumption satisfies $u'(c_t) = \xi_t$ for all t, where the co-state variable, ξ_t, solves $\dot{\xi}_t/\xi_t = \rho - r_t$. This gives the standard Euler equation:

$$g_t \equiv \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}, \tag{5}$$

where $-u''(c_t)c_t/u'(c_t) = \theta$ and g_t denotes the growth rate of consumption. From (5), there is a positive and linear relationship between the growth rate of the economy and the real interest rate. In balanced-growth equilibria where g_t remains constant, r_t does as well.

3.2 Capital good producers

Each capital producer chooses the quantity, $i_t \in \mathbb{R}_+$, of physical capital to produce at a unit cost in terms of the consumption good. The newly-formed capital is sold instantly in the capital market at the competitive price, p_K, expressed in terms of the final good. Hence, the problem of the capital producer at time t is

$$\max_{i_t \geq 0} \left\{ i_t(p_{K,t} - 1) \right\}.$$

For a solution with $i_t > 0$ to exist, $p_{K,t} = 1$ and profits are zero.

3.3 Valuation of firms

The financial value of a firm with productivity $j \in \{L, H\}$ and capital stock k is denoted $\upsilon_j(k)$. It is composed of all financial claims held by the creditors and shareholders of the firm. In this setting with perfectly efficient financial markets, the Modigliani-Miller theorem holds so that the value of the firm is independent from its capital structure.

Suppose the firm meets a broker. The post-trade capital stock, k', and the payment to the broker, ϕ, are determined by the generalized Nash bargaining solution where the broker's bargaining power is $\eta \in [0, 1]$. The firm's surplus from the negotiated outcome, (k', ϕ), is equal to its post-trade market value, $\upsilon_j(k')$, net of capital expenditure, $k' - k$, and payment to the broker, ϕ, minus the value of the firm if no trade takes place, $\upsilon_j(k)$. The surplus of the broker is simply ϕ. Hence, the outcome of the bargaining is given by:

$$\left[k_j, \phi_j(k)\right] \in \arg \max_{(k',\phi) \in \mathbb{R}_+^2} \left[\upsilon_j(k') - (k' - k) - \phi - \upsilon_j(k)\right]^{1-\eta} [\phi]^{\eta}. \tag{6}$$

The solution to (6) is

$$k_j \in \arg\max_{k' \geq 0} \{\upsilon_j(k') - k'\} \tag{7}$$

$$\phi_j(k) = \eta \max_{k' \geq 0} [\upsilon_j(k') - (k' - k) - \upsilon_j(k)], \quad j \in \{L, H\}. \tag{8}$$

From (7), the post-trade capital stock maximizes the value of the firm net of the investment in physical capital. It is the same choice that the firm would make if it had access to the capital market without a broker. From (8), the payment to the intermediary is a fraction, η, of the increase in the value of the firm net of the investment. Using the expression for ϕ_j in (8), the surplus of the firm can be reexpressed as

$$\upsilon_j(k_j) - (k_j - k) - \upsilon_j(k) - \phi_j(k) = (1 - \eta) \max_{k' \geq 0} [\upsilon_j(k') - (k' - k) - \upsilon_j(k)].$$

The firm receives a share, $1 - \eta$, of the surplus it would obtain if it had direct access to the capital market. Hence, from the standpoint of the firm, the economy where brokers have bargaining power, η, is equivalent to an economy where the firm accesses the capital market at rate $\alpha(1 - \eta)$ with full bargaining power. I make use of this equivalence in the following to write the Hamilton-Jacobi-Bellman (HJB) equation for υ_j.

The value of a firm with productivity state $j \in \{L, H\}$ and capital $k \in \mathbb{R}^+$ solves:

$$\begin{aligned} r\upsilon_j(k) &= A_j f(k) + \lambda \pi_{-j} [\upsilon_{-j}(k) - \upsilon_j(k)] \\ &\quad + \alpha(1 - \eta) \max_{k' \geq 0} [\upsilon_j(k') - (k' - k) - \upsilon_j(k)] + \dot{\upsilon}_j(k), \end{aligned} \tag{9}$$

where $-j \in \{L, H\} \backslash \{j\}$. The rate at which the firm discounts its profits is the rate of return of financial wealth in the household's problem. The left side of (9) is the opportunity cost from investing in the firm. On the right side, the firm generates an output flow equal to $A_j f(k)$. At Poisson arrival rate, $\lambda \pi_{-j}$, it draws a new productivity, $-j \neq j$. The capital gain is $\upsilon_{-j}(k) - \upsilon_j(k)$. At Poisson arrival rate, α, the firm has access to the capital market via a broker. The change in the value of the firm net of the investment is $\upsilon_j(k') - (k' - k) - \upsilon_j(k)$ and, after payment to the broker, the firm keeps a fraction $1 - \eta$ of that gain. The last term is the change in the value of the firm over time.

3.4 Tobin's q

From the assumption that $f_j(k) = A_j k$, it can be checked that $\upsilon_j(k) = q_j k$, where $q_j \in \mathbb{R}$ is the Tobin's q of the firm, i.e., it is the market value of a unit of physical capital invested in a firm with productivity j. The optimal choice of capital, from (7), can be reexpressed as

$$k_j \in \arg\max_{k' \geq 0} \{(q_j - 1) k'\}. \tag{10}$$

For this problem to have a solution, $q_j \leq 1$ for all $j \in \{L, H\}$. Firms might invest if $q_j = 1$ and they disinvest if $q_j < 1$. Substitute $\upsilon_j(k) = q_j k$ into (9) and divide both sides by k, the Tobin's q obeys the following dynamic equations:

$$rq_j = A_j + \lambda\pi_{-j}\left(q_{-j} - q_j\right) + \alpha(1-\eta)\left(1 - q_j\right) \quad \forall j \in \{L, H\}, \tag{11}$$

where $-j \in \{L, H\} \backslash \{j\}$. The left side is the opportunity cost of investing in the firm. The right side is the return from that investment, which is composed of the productivity, A_j, the expected capital gain as idiosyncratic productivity changes over time, and the gain from reselling the capital when the firm has access to the capital market. After some calculation,

$$q_j = \frac{\hat{A}_j + \alpha(1-\eta)}{r + \alpha(1-\eta)} \quad \forall j \in \{L, H\}, \text{ where } \hat{A}_j \equiv \frac{\left[r + \alpha(1-\eta)\right] A_j + \lambda \bar{A}}{r + \lambda + \alpha(1-\eta)}. \tag{12}$$

The Tobin's q has two components. The first term corresponds to a weighted average of the firm's current productivity, A_j, and its future expected productivity, $\bar{A}$, if it experiences a productivity shock. The second term is the expected discounted resale value of the firm's capital at the time the firm can access the capital market. In order for the capital market to be active, there must be a positive demand for capital. From (10) and (12), $\max_{j\in\{L,H\}}\{q_j\} = q_H = p_K = 1$.

Corollary 1 *A measure of the dispersion across Tobin's q is*

$$q_H - q_L = \frac{A_H - A_L}{r + \lambda + \alpha(1-\eta)}.$$

Frictions in the capital market generate a dispersion of Tobin's q across firms, which increases when search frictions become more severe, brokers have greater bargaining power, and idiosyncratic productivity shocks are more persistent.

3.5 Valuation of brokers

Let w_t denote the expected discounted sum of the profits of a broker. It solves:

$$rw_t = \alpha \int_{\{L,H\}\times\mathbb{R}^+} \phi_j(k) d\Upsilon_t(j, k) + \dot{w}_t, \tag{13}$$

where $\phi_j(k)$ is given by (8) and where $\Upsilon_t(j, k)$ is the Lebesgue-Stieltjes measure of firms across productivity types and holdings of capital. Brokerage firms are owned by households and their profits are discounted at rate r_t. According to (13), a broker meets a firm at random at rate α. From (8), the broker's revenue from this encounter is $\phi_j(k)$. From the solution to the bargaining problem, (8), in an equilibrium where $q_H = 1$, $\phi_H(k) = 0$, and

$\phi_L(k) = \eta(1-q_L)k$. Brokers only make profits when they meet firms that want to liquidate their capital. Using that $1-q_L = q_H - q_L$ and Corollary 1,

$$\phi_L(k) = \frac{\eta\left(A_H - A_L\right)}{r+\lambda+\alpha(1-\eta)}k. \tag{14}$$

I substitute the expression for $\phi_j(k)$ from (14) into (13) to obtain

$$rw_t = \alpha \int_{\mathbb{R}+} \phi_L(k) d\Upsilon_t(L,k) + \dot{w} = \frac{\alpha\eta\left(A_H - A_L\right)K_{L,t}}{r+\lambda+\alpha(1-\eta)} + \dot{w}_t, \tag{15}$$

where $K_{L,t} \equiv \int_{\mathbb{R}+} kd\Upsilon_t(L,k)$ is the aggregate capital stock held by low-productivity firms.

3.6 Distribution of capital

I assume that the distribution of firms across productivity levels at time 0 is given by the invariant distribution, $(\pi_j)_{j\in\{L,H\}}$. Moreover, I focus on equilibria where all type-H firms leave the capital market with the same amount of capital, $k_{H,t} = K_t/\pi_H$.

Under the assumption that $f_j(k)$ is linear, the distribution of capital across firms of the same type does not matter for allocative efficiency. However, the distribution of capital across firms of different productivities is critical. Let denote $K_H \equiv \int_{\mathbb{R}+} kd\Upsilon(H,k)$ the aggregate capital stock held by the most productive firms. It obeys the following law of motion:

$$\dot{K}_H = \alpha K_L + \lambda\pi_H K_L - \lambda\pi_L K_H + \dot{K}. \tag{16}$$

Over a small time interval of length dt, αdt firms access the capital market. A fraction π_L of those firms are type-L firms. Each low-productivity firm holds on average K_L/π_L. Hence, by the law of large number, a quantity $\alpha dt K_L$ of capital is reallocated from low-productivity firms to firms of type H.[18] According to the second term on the right side of (16), a measure λdt of firms receive a productivity shock and the realization is H with probability π_H. So, a quantity $\lambda dt \pi_H K_L$ of capital that was held by low-productivity firms is now held by firms of type H. Conversely, a measure $\lambda dt \pi_L$ of firms of type H receive a productivity shock with realization L. Hence, a quantity $\lambda dt \pi_L K_H$ of capital moves from firms H to firms L. The last term on the right side of (16) represents the accumulation of new capital that is allocated to the most productive firms, H.

I denote $\hat{k}_j = K_j/K$, $j \in \{L,H\}$, the share of the aggregate capital stock that is allocated to firms of type j. On a balanced-growth path, K_j grows at a constant rate g. Hence, from (16), using that $K_L = K - K_H$,

$$\hat{k}_H = 1 - \hat{k}_L = 1 - \frac{\lambda\pi_L}{\lambda+\alpha+g}. \tag{17}$$

[18] For the existence of a Law of Large Numbers for economies with a continuum of random variables, see Uhlig (1996).

A positive share of the capital stock, $\hat{k}_L$, is operated by the least productive firms. This share depends positively on the frequency of idiosyncratic productivity shocks, and negatively on the meeting rate with brokers and the growth rate of the economy.

Eisfeldt and Shi (2018) define capital reallocation "as the transfer or sale of capital between productive technologies or firms". Hence, in the model, capital reallocation relative to the aggregate capital stock is $\alpha\hat{k}_L$. Reallocation increases with α and λ but it decreases with g.

3.7 Taking stock

I define a balanced-growth equilibrium as a time-path for aggregate output, Y_t, aggregate consumption, C_t, aggregate capital, K_t, the distribution of the capital stock across productivity levels, $\{\hat{k}_j\}_{j\in\{L,H\}}$, Tobin's q, $\{q_j\}_{j\in\{L,H\}}$, the market capitalization of brokers, w_t, and the real interest rate, r, that satisfy the conditions below.

Aggregate output and TFP. The total output of the economy is equal to

$$Y_t = \overbrace{K_t \underbrace{\sum_{j\in\{L,H\}} A_j\hat{k}_j}_{\text{TFP}}}^{\text{value added}} = \overbrace{C_t + \dot{K}_t}^{\text{expenditure}}, \tag{18}$$

where $\hat{k}_j$ is given by (17). According to the right side of (18), total output is divided between aggregate consumption, C_t, and investment, $\dot{K}_t$. The term, $\sum_{j\in\{L,H\}} A_j\hat{k}_j$, is the endogenous total factor productivity of the aggregate production function that has capital as the sole input. By substituting $\hat{k}_j$ by its value given by (17), total factor productivity is equal to

$$TFP = A_H - \frac{\lambda\pi_L(A_H - A_L)}{\lambda + \alpha + g}. \tag{19}$$

The endogenous TFP is equal to the highest productivity, A_H, reduced by a misallocation term that depends on the market frictions, the idiosyncratic risk, and the endogenous growth rate of the economy.

Balanced growth path. Along a balanced growth path,

$$C_t = C_0 e^{gt}, \ \ K_t = K_0 e^{gt}, \tag{20}$$

where, from the Euler equation, (5), the growth rate of the economy is $g = (r-\rho)/\theta$, and $K_0 \in \mathbb{R}_+$ is given.

Aggregate financial wealth. Households' total financial wealth is defined as

$$\Omega_t \equiv w_t + \left(q_L \hat{k}_L + q_H \hat{k}_H\right) K_t, \tag{21}$$

where q_j is given by (12). The first term on the right side of (21) is the market capitalization of brokers while the second term is the market capitalization of firms. From (15),

$$(r-g)\, w_t = \frac{\alpha\eta(A_H - A_L)\lambda\pi_L}{(\lambda+\alpha+g)\left[r+\lambda+\alpha(1-\eta)\right]} K_t, \tag{22}$$

where I used the expression for $\hat{k}_L$ given by (17), and the restriction to balanced-growth equilibria where $\dot{w}_t = g w_t$. It is nonnegative and bounded if $r - g > 0$. From the budget identity in the household's problem, (4), using that $\dot{\omega}_t / \omega_t = g$,

$$C_0 = (r-g)\Omega_0. \tag{23}$$

The real interest rate. Setting $q_H = 1$ into (12), the real interest rate solves

$$r = A_H - \frac{\lambda\pi_L(A_H - A_L)}{r+\lambda+\alpha(1-\eta)}. \tag{24}$$

The equilibrium real interest rate is the unique positive solution to (24) and is represented graphically in Figure 2. It equals the productivity of type-H firms minus an *illiquidity discount*, which arises from difficulties in selling capital when productivity drops due to search and bargaining frictions. This discount increases with the idiosyncratic productivity risk, λ, the gap between high and low productivities, $A_H - A_L$, and brokers' bargaining power, η. It decreases with the rate at which firms meet brokers, α. In the Supplementary Appendix, I add frictions in the primary capital market by assuming that capital producers meets primary brokers with bargaining power η_0 at rate α_0. The real interest rate is determined as in (24) except that the left side is multipled by a term, $1 + r/\left[\alpha_0(1-\eta_0)\right]$, that depresses r further.

The comparison of (19) and (24) shows that r and average capital productivity (TFP) are distinct. The following Corollary provides a ranking.

Proposition 1 *Total factor productivity is greater than the rate of return of financial wealth,* $TFP > r$, *if and only if*

$$\alpha\eta > r - g. \tag{25}$$

The average productivity of capital is greater than the real interest rate when the product of the matching probability and the dealer's bargaining power is greater than the real interest

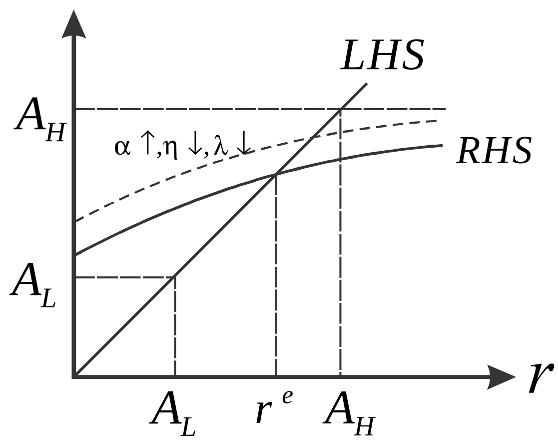

Figure 2: Determination of the equilibrium real interest rate.

rate net of the growth rate. Using that the right side of (25) is bounded above by A_H, the condition (25) holds, for example, when search frictions are small and dealers' bargaining power is not too low. This result can help explain why some countries experience low growth despite having a high TFP.

Proposition 2 ***(Existence and uniqueness of balanced-growth equilibrium.)*** *Suppose $K_{L,0} = \hat{k}_L K_0$. There exists a balanced-growth path equilibrium featuring $0 < g < r$ if and only if*

$$[\rho + \alpha(1-\eta)](A_H - \rho) + \lambda(\bar{A} - \rho) > 0, \tag{26}$$

and either $\theta \geq 1$ or

$$\frac{(1-\theta)A_H - \rho}{1-\theta} < \frac{(1-\theta)\,\lambda\pi_L(A_H - A_L)}{\rho + (1-\theta)\,[\lambda + \alpha(1-\eta)]}. \tag{27}$$

The equilibrium is unique and such that $r \in [\bar{A}, A_H]$. Moreover, $\partial r/\partial\alpha > 0$, $\partial r/\partial\eta < 0$, and $\partial r/\partial\lambda < 0$.

According to (26), a necessary and sufficient condition for the equilibrium growth rate to be positive is that ρ is less than a threshold, $\rho_0 \in (\bar{A}, A_H)$. A necessary condition is $A_H > \rho$ and a sufficient condition is $\bar{A} > \rho$. The lifetime utility of households is bounded if either $\theta \geq 1$ or if (27) holds, i.e., if $\rho/(1-\theta)$ is larger than some threshold. In the Supplementary Appendix, I characterize transitional dynamics for arbitrary initial conditions and show the equilibrium is unique.

3.8 Misallocation and growth

I now compare the equilibrium allocations to first-best allocations, where the planner is not subject to search frictions to reallocate capital, and to constrained-efficient allocations, where the planner is subject to the same search frictions as the ones faced by brokers and firms. I will also characterize allocations at the limit when search frictions vanish.

Misallocation relative to the first best A benevolent planner who can freely allocate capital across firms would only operate the H-technology, so that the level of output would be $Y_t^* = A_H K_t$, and it would accumulate capital at rate $g^* = (r^* - \rho)/\theta$, where $r^* = A_H$. The misallocation of capital relative to that first-best benchmark has both a *static* and a *dynamic* dimension.

The *static misallocation* refers to the loss of output relative to the first best taking K_t as given. It is equal to

$$MIS_Y \equiv \frac{Y_t^* - Y_t}{Y_t^*} = \hat{k}_L \left(\frac{A_H - A_L}{A_H} \right) = \frac{\lambda \pi_L}{\lambda + \alpha + g} \left(\frac{A_H - A_L}{A_H} \right). \tag{28}$$

From (19), $TFP = A_H (1 - MIS_Y)$. Misallocation increases with the idiosyncratic risk, λ, and with the search frictions, $1/\alpha$. The bargaining power of brokers has no direct effect on MIS_Y.[19] However, it has an indirect effect through g. As η increases, g decreases, and MIS_Y increases. It is due to the assumption that new capital is allocated instantly to H-firms while second-hand capital requires time for reallocation, $\hat{k}_L$ increases.

The *dynamic misallocation* refers to the rate of capital accumulation that is inefficiently low. I measure this misallocation by the difference in the growth rates between a frictionless economy and a frictional economy, i.e., $g^* - g$. It is equal to

$$g^* - g = \frac{r^* - r}{\theta} = \frac{\lambda \pi_L (A_H - A_L)}{\theta \left[r + \lambda + \alpha(1 - \eta) \right]},$$

where $r^* = A_H$. The right side has been obtained by substituting r by its expression given by (24). The *growth deficit*, $g^* - g$, is equal to the illiquidity discount of the real interest rate, which itself depends on r, multiplied by the intertemporal elasticity of substitution. It increases with η and λ but it decreases with α.

Table 1 indicates how MIS_Y and g co-vary in response to changes in the market structure and the stochastic process driving idiosyncratic productivity shocks. Irrespective of the

[19] If there are more than two realizations for the productivity shocks and if the stochastic process for these shocks obeys a continous-time Markov chain, then it is possible for brokers' bargaining power to affect the static misallocation of capital. See the Appendix.

underlying change affecting the economy, a greater misallocation of K_t is always associated with a lower growth rate.

	α	η	λ	π_L	A_L
MIS_Y	$-$	$+$	$+$	$+$	$-$
g	$+$	$-$	$-$	$-$	$+$

Table 1. Misallocation and growth

Misallocation relative to constrained-efficient allocations I now consider the problem of a social planner who is subject to the same trading frictions as the ones faced by firms and brokers. More specifically, over a time interval of length dt, the amount of capital that can be reallocated between low- and high-productivity firms is $\alpha dt K_t$. The planner chooses how to divide total output between consumption and capital formation and how to allocate the flow of capital that reaches the market across different firms.

Proposition 3 ***(Optimal growth.)*** *Suppose the initial condition is such that* $K_{L,0}/K_0 = \lambda\pi_L/(g+\alpha+\lambda)$. *The decentralized equilibrium is constrained-efficient if and only if* $\eta = 0$.

The presence of search frictions in the capital market is a technological constraint that cannot be undone by the social planner. However, the planner would like to eliminate the bargaining power of brokers. For a given K, the bargaining power of dealers does not generate static misallocation of capital. Indeed, the capital that reaches the market flows toward the most productive firms, irrespective of η. However, by capturing a fraction of the surplus generated by the reallocation of capital, brokers lower the rate of return of financial assets, the accumulation of financial wealth by households, and hence the rate of growth of the economy.[20]

Misallocation at the frictionless limit With the rise of the Internet and online platforms for advertising capital goods, search frictions in the capital market have significantly decreased. To reflect these advancements, I analyze the limit of the decentralized equilibrium as trading frictions in the capital market vanish ($\alpha \to +\infty$) and assess whether the equilibrium allocation converges to a first-best allocation. I distinguish two cases: firms have some bargaining power, $\eta < 1$, and brokers have all the bargaining power, $\eta = 1$.

[20] In Lagos and Rocheteau (2009), where both the real interest rate and the asset supply are exogenous, if the payoff from holding the asset is linear in asset holdings and preference shocks are i.i.d., then η does not affect welfare. In my model, however, it influences welfare through the growth rate of the capital stock. Under free entry of dealers, as in Lagos and Rocheteau (2007), constrained efficiency would require the Hosios condition to hold. I introduce free entry in Section 5.

Proposition 4 ***(Growth at the frictionless limit.)***

1. *Nonmonopolistic dealers, $\eta < 1$. As $\alpha \to +\infty$, $r \to A_H$ and $g \to (A_H - \rho)/\theta$.*

   - $\hat{k}_H \to 1$, $MIS_Y \to 0$, *and* $TFP \to A_H$.
   - $\phi_L(k) \to 0$ *for all* $k > 0$.

2. *Monopolistic dealers, $\eta = 1$. For all α, $r < A_H$ is the unique solution to*

$$r = A_H - \frac{\lambda}{r + \lambda}(A_H - \bar{A}), \tag{29}$$

   and $g = (r - \rho)/\theta < (A_H - \rho)/\theta$.

   - $\hat{k}_H \to 1$, $MIS_Y \to 0$, *and* $TFP \to A_H$.
   - $\phi_L(k) = (A_H - A_L)\, k/(r + \lambda) > 0$ *for all* $k > 0$.

When firms have some bargaining power, the frictionless limit of the decentralized equilibrium coincides with the textbook AK model. The real interest rate approaches the highest productivity, which then determines the rate of growth of the economy. All the capital is held by the most productive firms and payments to brokers vanish.

If brokers have all the bargaining power, then the equilibrium allocation fails to converge to a first best. This result is a version of the Diamond (1971) paradox whereby competitive pressures are inexistent, irrespective of the frequency at which agents meet each other, when one side of the market has all the bargaining power. Indeed, the effective rate at which firms meet brokers is $\alpha(1 - \eta) = 0$ for all $\alpha > 0$. It is easy to check from (29) that $r < A_H$ with $r \to A_H$ as $\lambda \to 0$ and $r \to \bar{A}$ as $\lambda \to +\infty$. It is only when the idiosyncratic productivity risk disappears that the economy converges to the competitive benchmark.

This last case is also useful to illustrate the distinction between static and dynamic misallocation. When $\eta = 1$, as α tends to infinity, the capital is optimally allocated in a static sense since it is held by the most productive firms and the average productivity of capital is maximum and equal to A_H. From (19), $TFP = A_H$, and, from (28), $MIS_Y = 0$. However, because brokers have all the bargaining power, the rate of return of financial wealth is lower than A_H. By investing in financial wealth, households cannot appropriate the full return of the capital stock they help finance. As a result, the economy grows at a lower rate than the Walrasian benchmark

4 Frictional capital market with knowledge spillover

The benchmark model with linear technologies studied so far reflects the conventional wisdom that the same market frictions contributing to capital misallocation also reduce the economy's long-run growth rate. So far, however, the model does not incorporate realistic diminishing returns to capital at the firm level and omits a key insight from the literature—namely, that growth arises from the production of knowledge. In the following, I generalize the technology to address both limitations.

Since the focus of this paper is on the capital market, I follow Arrow (1962), Frankel (1962), and Romer (1986), and assume that knowledge is created by learning how to operate capital. As firms install new machines, they learn the most efficient ways to operate them.[21] For instance, today, investments in artificial intelligence technologies generate substantial learning-by-doing and knowledge spillover (e.g., Chen, Shi, and Srinivasan, 2024). The new knowledge, which is nonrival, is accessible to all firms. In Arrow's words:

> "[I take] cumulative gross investment (cumulative production of capital goods) as an index of experience. Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli."

Formally, the technology of a firm is of the form $A_j K^{1-\beta} k^{\beta}$ where $\beta \in (0, 1)$ and K is the cumulative production of capital goods. While the technology features diminishing marginal returns with respect to k, it has constant returns to scale with respect to both K and k. In the spirit of Arrow (1962), K enters total factor productivity because it is a measure of the stock of knowledge or experience.[22] When households finance the production of new capital through their financial wealth accumulation, they do not take into account how it affects the stock of knowledge and firms' productivity. This knowledge externality

[21] Rosenberg (1963) argues that "the machinery-producing industries possess certain unique characteristics which played a major role in accounting for the rapid production and diffusion of technological innovations." De Long and Summers (1991, 1992) and De Long (1992) document the strong nexus between equipment investment and economic growth. They find that equipment has a high net social return, about 20 percent per year. Irwin and Klenow (1994) provide evidence of learning-by-doing spillovers in the semiconductor industry. The role of investment externalities for long-term growth has been challenged by Benhabib and Jovanovic (1991) and Jones (1995).

[22] In Arrow (1962) and Jovanovic and Rousseau (2002), technological change is embodied in capital goods. In the Supplementary Appendix, I consider such a version of the model and show that the results are robust. For a presentation and discussion of Arrow's model, see also Solow (1997). For a model of learning by doing and technology adoption, see Jovanovic and Nyarko (1996). For a Schumperian growth model where both capital accumulation and innovation contribute to long-run growth, see Howitt and Aghion (1998). In the Supplementary Appendix, I consider an alternative version of my model in which knowledge is associated with an input factor other than capital, but it complements capital in production.

is stronger as β decreases. In the previous section, $\beta = 1$ and the externality is mute. I generalize the set of idiosyncratic productivities to any arbitrary finite set, $\{A_1, ..., A_j, ...A_J\}$, with $0 < A_1 < ... < A_j < ... < A_J$ and $\bar{A} \equiv \sum_{j=1}^{J} \pi_j A_j$. The following Lemma characterizes the first-best allocation chosen by a planner who is not subject to any friction to allocate the capital among firms.

Lemma 1 ***(First-best allocations.)*** *Assume*

$$1 - \theta < \frac{\rho}{\left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}} < 1. \tag{30}$$

At the first best, the economy grows at rate $g^* = (r^* - \rho)/\theta$ *where*

$$r^* = \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}. \tag{31}$$

The share of capital allocated to firms of type j, $\pi_j k_{j,t}/K_t$, *is*

$$\Xi_j^* \equiv \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}} = \pi_j \left(\frac{A_j}{r^*}\right)^{\frac{1}{1-\beta}} \quad \forall j \in \{1, ..., J\}. \tag{32}$$

As shown from (32), and in contrast to the linear technology studied earlier, provided $\beta < 1$, it is optimal to allocate a positive amount of capital to all firms such that $A_j > 0$.

4.1 Firms' valuations

The value of a firm solves

$$\begin{aligned} r\upsilon_{j,t}(k) &= A_j K_t^{1-\beta} k^\beta + \lambda \sum_{x=1}^{J} \pi_x \left[\upsilon_{x,t}(k) - \upsilon_{j,t}(k)\right] \\ &\quad + \alpha(1-\eta) \max_{k' \geq 0} \left[\upsilon_{j,t}(k') - (k' - k) - \upsilon_{j,t}(k)\right] + \dot{\upsilon}_{j,t}(k), \end{aligned} \tag{33}$$

for $j \in \{1, ..., J\}$. Relative to the previous section, $\upsilon_{j,t}(k)$ is not time-invariant because productivity, which depends on K_t, grows over time. On a balanced-growth path, the output flow, $A_j K_t^{1-\beta} k^\beta = A_j e^{(1-\beta)gt} K_0^{1-\beta} k^\beta$, grows at rate $(1-\beta)g$. Using this observation, the following lemma provides a closed-form solution for the value function.

Lemma 2 ***(Firms' valuations.)*** *The value function of the firm is*

$$\upsilon_{j,t}(k) = \frac{\hat{A}_j K_t^{1-\beta} k^\beta}{r - (1-\beta)g + \alpha(1-\eta)} + \frac{\alpha(1-\eta)}{\alpha(1-\eta) + r} k + \Lambda_{j,t}, \tag{34}$$

where $\Lambda_{j,t} \in \mathbb{R}$ *is independent of* k *and*

$$\hat{A}_j \equiv \frac{[r-(1-\beta)g+\alpha(1-\eta)]\,A_j+\lambda\bar{A}}{r-(1-\beta)g+\alpha(1-\eta)+\lambda}. \tag{35}$$

The first term on the right side of (34) is the expected discounted sum of output flows from t until the next effective access to the capital market at Poisson arrival rate, $\alpha(1-\eta)$. The effective discount rate on the denominator, $r-(1-\beta)g+\alpha(1-\eta)$, accounts for the growth rate of productivity, $(1-\beta)g$, in accordance with the Gordon (1959) growth formula. The second term on the right side is the expected discounted resale value of the capital at the time of the next effective access to the market. The effective discount rate of that second term, $\alpha(1-\eta)+r$, does not incorporate the growth rate of productivity. The reason is that while the output flow of the firm is growing over time, the price of capital remains constant and equal to $p_K = 1$. The third term is the continuation value that is independent of k. The quantity, $\hat{A}_j$, given by (35) generalizes the expression in (12). It is equal to a weighted average of the current productivity, A_j, and the future expected productivity, $\bar{A}$, where the weight on A_j decreases with g.

The problem of the firm of type $j \in \{1, ..., J\}$ who has access to the capital market at time t is $\max_{k\geq 0}\{-k+v_{j,t}(k)\}$. By substituting $v_{j,t}(k)$ by its expression given by (34), it can be rewritten as:

$$k_{j,t} = \arg\max_{k\geq 0}\left\{-rk+\left[\frac{r+\alpha(1-\eta)}{r+\alpha(1-\eta)-(1-\beta)g}\right]\hat{A}_j K_t^{1-\beta}k^{\beta}\right\}. \tag{36}$$

The novelty relative to the problem of a firm in a frictionless market is the term between squared brackets on the right side. This term, which is greater than one, captures the fact that the firm can only readjust its capital infrequently, at effective rate $\alpha(1-\eta)$, while productivity is growing at rate $(1-\beta)g$. The first-order condition is

$$\tilde{k}_j^{1-\beta} \equiv \left(\frac{k_{j,t}}{K_t}\right)^{1-\beta} = \frac{\beta\,[r+\alpha(1-\eta)]}{r\,[r+\alpha(1-\eta)-(1-\beta)g]}\hat{A}_j, \tag{37}$$

for $j \in \{1, ..., J\}$. Provided $r > (1-\beta)g-\alpha(1-\eta)$, the demand for capital is positive. It increases with the effective productivity of the firm, $\hat{A}_j$, which itself depends on market structure via α and η. Productivity growth, $(1-\beta)g$, affects the aggregate demand for capital through two channels. First, there is the capitalization effect that reduces the effective discount rate and tends to increase the demand for capital. Second, productivity growth generates a mean-preserving decrease in the spread of $\{\hat{A}_j\}$, which reduces the aggregate demand for capital.

4.2 Distribution of firms

The capital of a firm is determined by the last time it accessed the capital market, $t-\tau$, and its productivity at that time, $i \in \{1, ..., J\}$. In order to determine the balanced-growth distribution of firms, I allow $t-\tau$ to be negative, i.e., a firm's last access to the market can be arbitrarily far in the past. Given (i, τ), the firm's capital corresponds to its optimal choice the last time it accessed the market at time $t-\tau$, $k_{i,t-\tau}$. Here, $k_{i,t-\tau} = e^{-g\tau} k_{i,t}$ where $k_{i,t}$ is defined by (36) and is proportional to K_t, thereby growing at rate g. Therefore, the state of a firm at time t—its capital and its current productivity—can be identified with the triplet, $(i, j, \tau) \in \{1, ..., J\}^2 \times \mathbb{R}^+$. The following lemma provides a closed-form expression for the distribution of firms across states (i, j, τ).

Lemma 3 ***(Distribution of firms.)*** *Let $\gamma(i, j, \tau)$ denote the density measure of firms of type $(i, j, \tau) \in \{1, ..., J\}^2 \times \mathbb{R}^+$. It is equal to*

$$\gamma(i, j, \tau) = \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau}) \pi_j \right]. \tag{38}$$

The density measure of firms who contacted the market at time $t-\tau$ and were of type i at that time is $\pi_i \alpha e^{-\alpha\tau}$, where I used the independence between the firm-broker matching process and the process for idiosyncratic productivity shocks. The density of the former is an exponential distribution with parameter α while the distribution of the idiosyncratic productivities is $\{\pi_i\}_{i=1}^J$. If the current type of a firm is $j \neq i$, then the firm had to experience at least one productivity shock over the time interval $(t-\tau, t)$, with probability $1 - e^{-\lambda\tau}$, and the realization of the last shock was j with probability π_j. If the current type is $j = i$, it is also possible that the firm did not experience any shock, with probability $e^{-\lambda\tau}$.

The total capital held by firms with idiosyncratic productivity j is

$$K_{j,t} \equiv \sum_{i \in \{1,...,J\}} \int_{\mathbb{R}^+} k_{i,t-\tau} \gamma(i, j, \tau) d\tau, \quad \forall j \in \{1, ..., J\}. \tag{39}$$

In the following lemma, $K_{j,t}$ is computed in closed form.

Lemma 4 ***(Distribution of capital.)*** *In a balanced-growth path equilibrium, the share of the aggregate capital stock held by type-j firms is equal to*

$$\frac{K_j}{K} = \frac{\pi_j}{g + \alpha + \lambda} \left[\alpha \left\{ \frac{\beta [\alpha(1-\eta) + r]}{r [r - (1-\beta)g + \alpha(1-\eta)]} \right\}^{\frac{1}{1-\beta}} \hat{A}_j^{\frac{1}{1-\beta}} + \lambda \right]. \tag{40}$$

4.3 Equilibrium real interest rate

I close the characterization of the equilibrium with the determination of r.[23] Market clearing requires that

$$\alpha \sum_{j=1}^{J} \pi_j k_{j,t} = (\alpha + g) K_t. \tag{41}$$

The left side represents the demand of capital emanating from firms accessing the market at rate α. The measure π_j of firms with productivity A_j demand $k_{j,t}$. The right side corresponds to the supply of capital. By the law of large numbers, a firm who accesses the market holds on average the mean of the distribution of capital across firms, K_t. In addition, capital producers supply $\dot{K}_t = gK_t$ units of new capital per unit of time.

Substitute $k_{j,t}$ by its value in (41) to show that the real interest rate solves

$$r = \underbrace{\frac{[\alpha(1-\eta)+r]}{r-(1-\beta)g+\alpha(1-\eta)}}_{\text{Gordon capitalization effect}} \times \underbrace{\left(\frac{\alpha}{\alpha+g}\right)^{1-\beta}}_{\text{congestion effect}} \times \underbrace{\beta}_{\text{knowledge externality}} \times \hat{r}^*, \tag{42}$$

where

$$\hat{r}^* \equiv \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{1-\beta}.$$

I now compare the expression for r given by (42) with the first-best value, r^*, given by (31). The last term on the right side of (42), $\hat{r}^*$, is a CES aggregator of individual productivities similar to r^*. In the decentralized equilibrium, $\hat{r}^*$ is constructed from the effective productivities, $\hat{A}_j$, instead of the current ones, A_j. These effective productivities are endogenous and depend on r. Using that $\{(\hat{A}_j, \pi_j)\}_{j=1}^J$ is a mean-preserving reduction of the spread of $\{(A_j, \pi_j)\}_{j=1}^J$, together with the observation that $x^{\frac{1}{1-\beta}}$ is a convex function, $\hat{r}^* < r^*$.

The term, β, scales down $\hat{r}^*$ because firms and households do not internalize the effect of their capital and financial wealth accumulation decisions on aggregate productivity. The share, β, is the elasticity of the firm's output with respect to its own capital.

The second term on the right side of (42), $[\alpha/(\alpha+g)]^{1-\beta}$, is a congestion effect that captures the negative pressure that the flow of new capital, g, exerts on the real interest rate. Indeed, in the presence of trading frictions, only a flow of firms of measure αdt accesses the market over an infinitesimal time interval of length dt. As a result, the flow of new capital goods produced during that time interval, gdt, raises the supply relative to the demand, which tends to reduce r.

[23] In the supplementary appendix, I describe a different but equivalent approach to market clearing according to which the mean of the distribution of capital holdings across all firms must be equal to the aggregate capital stock, K_t.

Finally, the first term on the right side of (42) is a capitalization effect, akin to the one in Gordon's formula, that scales the real interest rate up. It accounts for the fact that the productivity of capital is growing between two consecutive accesses to the market.

If the primary capital was frictional, where capital producers meet primary brokers with bargaining power η_0 at rate α_0 (see Supplementary Appendix), then the price of capital in the inter-broker market would be equal to $p_K = 1 + r/\left[\alpha_0(1-\eta_0)\right]$, which would tend to depress capital demands by firms and push the real interest rate downward.

In any equilibrium, r is a solution to (42) where $g = (r-\rho)/\theta$. It must satisfy $r > \rho$, so that the growth rate is positive, and $r > g$, i.e., $r(1-\theta) < \rho$, so that the representative household's lifetime utility is finite. The equilibrium has a simple recursive structure. The real interest rate is determined by (42), which gives the growth rate of the capital stock, g. Given r, (37) determines the demand for capital and (38) gives the distribution of capital across firms. The following proposition provides sufficient conditions for the existence of an equilibrium.

Proposition 5 ***(Existence of equilibria with capital market frictions and learning externality.)*** *If*

$$\rho < \beta \left\{ \sum_{j=1}^{J} \pi_j \left\{ \frac{\left[\rho + \alpha(1-\eta)\right] A_j + \lambda \bar{A}}{\rho + \alpha(1-\eta) + \lambda} \right\}^{\frac{1}{1-\beta}} \right\}^{1-\beta}, \tag{43}$$

and either $\theta \geq 1$or $\theta < 1$ and

$$\frac{\rho}{1-\theta} > \left[1 - \frac{(1-\beta)\alpha\left(1-\theta\right)\left(1-\eta\right)}{\beta\rho + \alpha\left(1-\theta\right)\left(1-\eta\right)}\right] \left\{ \frac{\alpha\left(1-\theta\right)}{\alpha\left(1-\theta\right)+\rho} \sum_{j=1}^{J} \pi_j \left[\hat{A}_j\left(\frac{\rho}{1-\theta}\right)\right]^{\frac{1}{1-\beta}} \right\}^{1-\beta}, \tag{44}$$

then there exists a balanced-growth equilibrium. The equilibrium real interest rate is bounded above by r^* given by (31).

According to (43), if $\theta \geq 1$, an equilibrium with positive growth exists provided that the rate of time preference is lower than a linearly homogeneous function of firms' productivities. If $\theta < 1$, an additional condition, (44), guarantees that the lifetime utility of households is bounded for all $t < +\infty$.

4.4 Limiting cases

I study several limiting cases in order to disentangle the roles played by the knowledge externality, market frictions, and the idiosyncratic productivity risk in the determination of the real interest rate and the allocation of capital across heterogenous firms.

Proposition 6 *Consider the following limiting cases.*

1. *Vanishing learning externality. As $\beta \to 1$,*

$$r \to \lim_{\beta \to 1} \hat{A}_J = \frac{\{r + \alpha(1-\eta)\} A_J + \lambda \bar{A}}{r + \alpha(1-\eta) + \lambda}. \tag{45}$$

The share of capital allocated to firms of type $j \in \{1, ..., J\}$ is

$$\frac{K_j}{K} = \frac{g + \alpha}{g + \alpha + \lambda} \mathbb{I}_{\{j=J\}} + \frac{\lambda \pi_j}{g + \alpha + \lambda}. \tag{46}$$

2. *Frictionless limit: Non-monopolistic brokers. Suppose $\eta < 1$. As $\alpha \to +\infty$,*

$$r \to \beta \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta} = \beta r^*. \tag{47}$$

The share of capital allocated to firms of type $j \in \{1, ..., J\}$ is

$$\frac{K_j}{K} = \Xi_j^* \equiv \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}}. \tag{48}$$

3. *Frictionless limit: Monopolistic brokers. Suppose $\eta = 1$. As $\alpha \to +\infty$, r tends to a solution to*

$$r - \rho = \frac{\theta}{\theta - (1-\beta)} \left[\beta \left\{ \sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r) \right]^{\frac{1}{1-\beta}} \right\}^{1-\beta} - \rho \right], \tag{49}$$

where

$$\hat{A}_j(r) \equiv \frac{\{r[\theta - (1-\beta)] + (1-\beta)\rho\} A_j + \theta \lambda \bar{A}}{r[\theta - (1-\beta)] + (1-\beta)\rho + \theta \lambda}. \tag{50}$$

The share of capital allocated to firms of type $j \in \{1, ..., J\}$ is

$$\frac{K_j}{K} = \frac{\pi_j \hat{A}_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}}. \tag{51}$$

4. *Vanishing idiosyncratic risk. As $A_j \to \bar{A}$ for all $j \in \{1, ..., J\}$,*

$$r \to \underbrace{\frac{\alpha(1-\eta) + r}{\alpha(1-\eta) + r - (1-\beta) g}}_{\text{Gordon effect}} \times \underbrace{\left(\frac{\alpha}{\alpha + g} \right)^{1-\beta}}_{\text{congestion effect}} \times \underbrace{\beta \bar{A}}_{\text{frictionless rate}}. \tag{52}$$

The first part of Proposition 6 shows that the equilibrium when $\beta \to 1$ coincides with the one from Section 3. As the knowledge externality vanishes, the real interest rate approaches $\max_j\{\hat{A}_j\}$. According to (46), some capital is misallocated in that it is held by less productive firms. The first term on the right side represents the (re)allocation of capital towards the most productive firms – those of type J. It is composed of the new capital produced at rate g and the existing capital that returns to the market at rate α. The second term represents the idiosyncratic shocks that redistribute the capital randomly across productivity types at rate λ.

According to the second part of Proposition 6, as trading frictions vanish, the real interest rate approaches the product of $r^*$and β. This result is a generalization of Romer (1986) in an environment with heterogenous firms. Firms do not internalize the effect of their capital choice on the creation of knowledge. As a result, firms underestimate the rate of return of their investment in capital goods. However, from (48), the share of K_t that is allocated to firms of type j coincides with the capital share in the first-best allocation.

The third part of Proposition 6 also considers the limit as search frictions vanish but it assumes that brokers are perfect monopolists with full bargaining power and information. It can be seen from (49) that

$$r > \beta \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{1-\beta}.$$

The right side corresponds to the Walrasian real rate in (47) where A_j has been replaced with $\hat{A}_j$. Thus, in general, r does not converge to the Walrasian real interest rate, βr^*. However, it is not necessarily the case that r is less than βr^* at the limit. Indeed, since $\hat{A}_j \to A_j$ as $\lambda \to 0$, it is possible for r to be greater than the Walrasian rate, βr^*, when the idiosyncratic risk is small. I will return to this result later. From (51), the share of capital operated by firms of type j has a similar expression as the first-best share, (32), where A_j has been replaced with $\hat{A}_j$. So, even though search frictions vanish, capital remains misallocated.

The fourth part of Proposition 6 gives the expression for r in the absence of idiosyncratic productivity shocks, when $A_j \to \bar{A}$ for all j. Frictions in the capital market are relevant, as shown by the first two terms in (52), because, in the presence of a growing TFP, firms would like to readjust their capital stock continuously over time. The competitive real interest rate, $\beta\bar{A}$, is multiplied by two factors. The first factor corresponds to a capitalization effect. This term is greater than one because productivity growth induces firms to front-load their demand for capital. The second term is a congestion term that is less than one because limited participation of firms in the market tends to depress demand.

4.5 Misallocation

In accordance with Section 3, I distinguish two forms of misallocation.

Dynamic misallocation From Proposition 5, $r < r^*$, which implies that the equilibrium rate of capital accumulation is lower than that in the first best. This inefficiency persists in a frictionless economy. Indeed, from Proposition 6, the equilibrium real interest rate in a Walrasian economy is $\beta r^* < r^*$. Relative to the case where there is no learning externality, the real interest rate does not always increase when market frictions are reduced and it can be larger than the frictionless value.

Let's start by explaining why market frictions have an ambiguous effect on r. I mute the congestion term in (42) by considering the case where α is large relative to g and I rewrite (42) as

$$r \approx \left\{1 - \frac{(1-\beta)(r-\rho)}{\theta\left[\alpha(1-\eta)+r\right]}\right\}^{-1} \beta \left\{\sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r;\alpha,\eta)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta}, \tag{53}$$

where I used that $\alpha\theta/(\alpha\theta + r - \rho) \to 1$. Consider an increase in η. Whenever a firm accesses the market for capital goods, it anticipates the holdup problem it will face in the future when readjusting its capital. There are two distinct considerations when responding to this holdup problem. First, firms hedge against future idiosyncratic shocks by taking a position that is closer to the average value for A_j. Formally, as η increases, $\hat{A}_j$ increases if $A_j < \bar{A}$ and it decreases if $A_j > \bar{A}$. Thus, an increase in η generates a mean-preserving reduction in the distribution of effective productivities, $\hat{A}_j$. From (53), using that $\hat{A}_j^{\frac{1}{1-\beta}}$ is a convex function of $\hat{A}_j$, the reduction in the dispersion of effective productivities pushes r downward.

Second, firms anticipate that they will demand more capital in the future, for a given A_j, due to a growing aggregate productivity. They mitigate the associated holdup problem by frontloading their demand for capital. This effect, which is captured by the first term on the right side of (53), contributes to a higher real interest rate. When the idiosyncratic productivities, A_j, are very concentrated, the second effect can dominate.

The following Proposition establishes that r can be larger than its Walrasian value for the case when the distribution of productivities is degenerate.

Proposition 7 ***(Holdup and real interest rate.)*** *Suppose $A_j = \bar{A}$ for all $j \in \{1, ..., J\}$. Moreover, $\eta = 1$ and $\beta \in (1-\theta, 1)$. There exists a $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$, the equilibrium real interest rate is greater than the Walrasian real interest rate, i.e., $r > \beta\bar{A}$.*

If brokers have all the bargaining power and the capital market is not too illiquid, then the real interest rate is greater than the one in a Walrasian economy. So, a high bargaining power reduces the dynamic misallocation. This result may initially seem surprising, as one might expect that when brokers are able to extract the entire surplus from their trades with firms, the rate of return on financial wealth would decline, moving even farther from the first-best outcome. However, it is important to recall that this result arises under the assumption that capital producers can trade competitively in the interbroker market and, furthermore, there is no static inefficiency when firms trade with brokers—the main inefficiency manifests itself through an intertemporal holdup problem. As a result, the incentives to produce capital remain present. The next corollary shows that reducing brokers' bargaining power from $\eta = 1$ can reduce households' welfare.

Corollary 2 *Suppose $\theta > 1$, $A_j = \bar{A} > \rho$ for all j, and $\alpha \to +\infty$. Initially, brokers are perfect monopolists, $\eta = 1$, so that $r > \beta A$. If η falls unexpectedly below one, then the representative household is worse off.*

In Corollary 2, I focused on the case where search frictions are asymptotically non-existent so that there are no transitional dynamics to take into account in the welfare calculation. In the absence of idiosyncratic risk, households can be made better off by allocating more bargaining power to brokers. However, there is a social cost when the distribution of A_j is nondegenerate: capital is not distributed so as to maximize output.

Capital misallocation. From (40) and (64), the share of K_t that is allocated to firms of type j is

$$\frac{K_j}{K} = \hat{\Xi}_j + \frac{\lambda}{g+\alpha+\lambda}\left(\pi_j - \hat{\Xi}_j\right) \quad \text{where } \hat{\Xi}_j \equiv \frac{\pi_j \hat{A}_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}}. \tag{54}$$

The first term on the right side of (54), $\hat{\Xi}_j$, describes the allocation of the capital among the firms accessing the market at a given point in time. The share of capital received by type-j firms is analogous to the first-best share, Ξ_j^*, defined in (48), where current productivity, A_j, is replaced with effective productivity, $\hat{A}_j$. The second term on the right side of (54) corresponds to the misallocation generated by the occurrence of idiosyncratic productivity shocks. Those shocks allocate the capital to type-j firms according to the shares, π_j.

Proposition 8 ***(Capital misallocation.)*** *In equilibrium, the share of capital allocated to the least productive firms is too high, while the share of capital allocated to the most productive firms is too low.*

I now measure the degree of overall capital misallocation by the loss of output for a given capital stock, K_t, relative to the first best. At the first-best allocation, the average output per unit of capital is

$$TFP^* \equiv \frac{Y_t}{K_t} = \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta}. \tag{55}$$

It is the maximum output that can be obtained from a given capital stock, K. In equilibrium, the output per unit of capital is

$$TFP \equiv \frac{Y_t}{K_t} = \int_0^{+\infty} \sum_{(i,j)\in\{1,\dots,J\}^2} A_j e^{-\beta g \tau} \tilde{k}_i^{\beta} \gamma(i,j,\tau) d\tau, \tag{56}$$

where $\tilde{k}_i \equiv k_{i,t}/K_t$ is given by (37). The right side of (56) sums the output of all firms across the distribution of states, $\gamma(i,j,\tau)$. The output of a firm in state (i,j,τ) normalized by K_t is $A_j \left(k_{i,t-\tau}/K_t\right)^{\beta} = A_j e^{-\beta g \tau} \tilde{k}_i^{\beta}$.

Proposition 9 ***(Static misallocation.)*** *If α and λ are large and if $\eta = 0$, then*

$$TFP \approx \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{1-\beta}. \tag{57}$$

A decrease in α or an increase in λ generate an increase in $MIS_Y \equiv \left(TFP^* - TFP\right)/TFP^*$.

4.6 A calibrated example

I illustrate the results above with a calibrated example. The parameterization of the model is summarized in Table 2. The unit of time is a year. I set $\rho = 0.01$ and $\theta = 2$. I use the average productivity, $\bar{A}$, to target $g = 2\%$. The technology is $A_j K^{1-\beta} k^{\beta}$ where $\beta = 0.4$. The process for the idiosyncratic productivity shocks is analogous to the one in Cui (2022). I target a measure of capital reallocation of 2% of firms' total assets annually (Eistfeldt and Shi, 2018), which gives $\eta = 0.93$. I set $\alpha = 2$ based on different sources for the turnover of machines and equipment. I will vary α and η to check the robustness of the results. Additional details are provided in the supplementary appendix.

Parameter	Explanation/Target	Value
ρ	rate of time preference	0.01
θ	intertemporal elasticity of substitution	2
β	capital share	0.4
(A_L, A_H)	Dispersion of firm productivities and growth rate	$(0.089, 0.146)$
(π_H, π_L)	Representation of US plant-level productivity as AR(1)	(0.5,0.5)
λ	Persistence of US plant-level productivity	0.5
α	Time to sell machinery equipment	2
η	Capital reallocation	0.926

Table 2. Calibrated parameter values

For these parameter values, the competitive real interest rate is $r^{ce} = 4.79\%$, which is less than the equilibrium value, $r = 5\%$, while the first-best rate of return of capital is $r^* = 12\,\%$.[24] The equilibrium growth rate, $g = 2\%$, is also larger than the one in the frictionless economy, $g^{ce} = 1.\,89\%$.

Real interest rate and market frictions The left panel of Figure 3 plots r as a function of α. The relation between r and α is monotonically increasing when $\eta = 0$ but it is decreasing over some range when the broker's bargaining power is high ($\eta = 0.93$ or $\eta = 0.99$). Thus, advances in trading technologies that reduce search frictions in the capital market do not necessarily lead to higher economic growth rates. If the market for capital goods is rendered frictionless, by taking the limit as α goes to $+\infty$, r falls by about 0.2 percentage points and g falls by about 0.4 percentage points. So, maybe surprisingly, making the market for capital goods more liquid hurts economic growth.

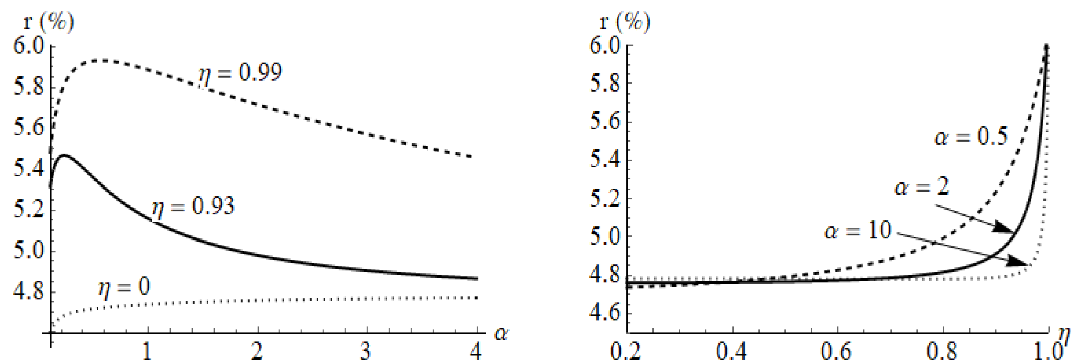

Figure 3: Real interest and capital market frictions

The right panel of Figure 3 shows r as a function of η. Over a large range of values for η, r increases with η, and this increase becomes particularly steep as η approaches one. For example, under the benchmark calibration, when η increases from 0.93 to 0.99, r rises by more than one percentage point. Even though the holdup problem is made more severe, the rate of return of financial claims, and hence the growth rate of the economy, increase.

Misallocation The top panels of Figure 4 depict the demand for capital by firms accessing the market, while the bottom panels report the shares of the total capital stock operated by firms with different levels of productivity. As trading frictions decrease, both the demands

[24] Setting $\beta = 0.4$ generates a strong learning-by-investing externality. As a result, there is a large difference between the competitive real interest rate and the first-best one. It is consistent with some studies, e.g., De Long and Summers (1992, p.159) "find that equipment appears to have a very high net social return-in the range of 20 percent per year; more than half of this comes from increased TFP".

and the shares become more disperse. The most productive firms demand more capital, except when α is very low, whereas less productive firms demand less. Conversely, as brokers' bargaining power increases, the demands and shares become more concentrated.

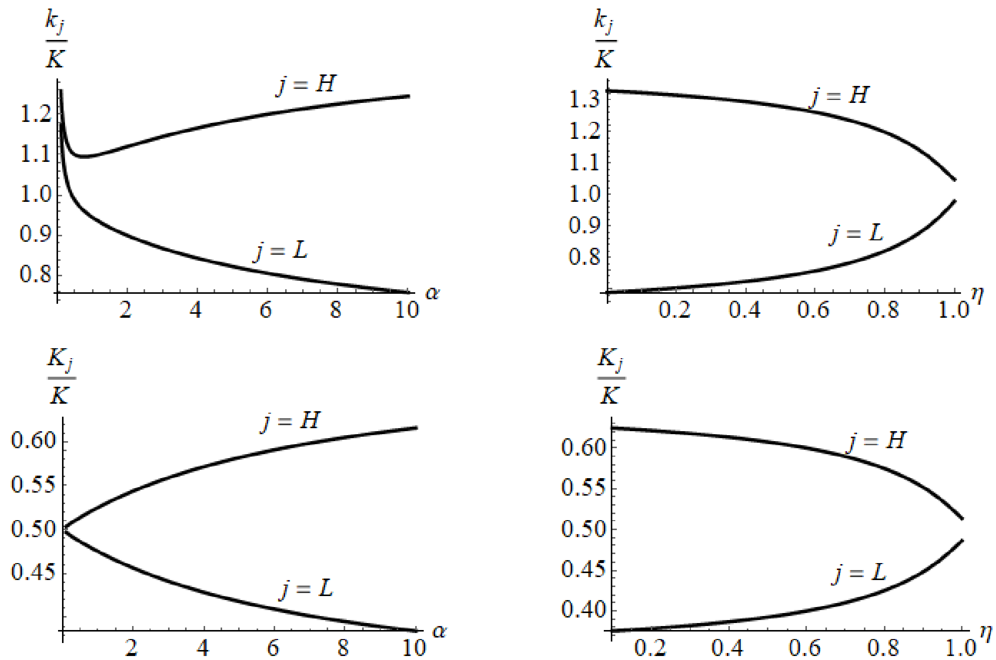

Figure 4: Top panels: demand for capital for firms of productivity type j; Bottom panels: share of total capital held by firms of productivity type j.

Figure 5 plots the measure, MIS_Y, of the output that is lost relative to the first-best allocation due to the misallocation of capital, taking K_t as given. For the benchmark calibration, the loss of output is about 1.2%. It increases to about 1.5% if α is divided by 2 and it falls to 0.85% if α is multiplied by 2. If η increases to 1, then the misallocation loss rises to 1.7%.

Welfare gain from eliminating market frictions I measure the welfare gain from removing market frictions by determining the fraction, Δ, of future consumption that households would be willing to forego to live in an economy with a frictionless capital market, where $\alpha(1-\eta) \to +\infty$, compared to an economy with a frictional capital market. (See the Appendix for a closed-form expression for Δ.) It should be noted that moving from an equilibrium with $\alpha \to +\infty$ involves no transition since capital can be reallocated instantly across firms.

Figure 6 plots Δ as a function of market frictions, α and η, for different values of β, which determine the magnitude of the learning externality. The plain curves correspond to the value of β used in the benchmark calibration, $\beta = 0.4$. The left panel shows that Δ is nonmonotone in α. It is negative when α is sufficiently low (but not too close to 0) and it is positive when α is above some threshold. Thus, removing market frictions is beneficial when the market is not too illiquid and it can be harmful otherwise. If η is set at its calibrated value, the α that maximizes household's lifetime utility along a balanced growth path is slightly above 0.2. In the right panel, the welfare gain from removing frictions is positive when η is below a threshold value and it is negative as η approaches one.

The dashed and dotted lines correspond to $\beta = 0.6$ and $\beta = 0.8$, respectively, so that the learning externality gets weaker.[25] In the left panel, Δ is positive for all α. So, if the learning externality is not sufficiently strong, eliminating market frictions is welfare improving. Moreover, the welfare gain can be sizeable. For instance, if $\beta = 0.8$ and α is less than one, then the welfare gain is greater than 10%. The right panel, where Δ is plotted against η, has a similar message.

5 Endogenous market structure and growth

I now study the relationship between misallocation and growth when the micro-structure of the capital market and misallocation are jointly determined. For this, I endogenize α, the speed at which firms can access brokers.[26] I interpret α as the outcome of a general matching function between firms and brokers and I allow for free entry of the latter. For this extension, I set $\beta = 1$ and return to the model of Section 3 in order to show that the knowledge externality is not needed to obtain a nonmonotone relationship between r and market frictions.

The rate at which a firm meets a broker is given by an increasing and strictly concave function, $\alpha(b)$, where b is the measure of brokers per firm. It is such that $\alpha(0) = 0$, $\alpha(+\infty) = +\infty$, $\alpha'(0) = +\infty$, and $\alpha'(+\infty) = 0$. Since meetings between firms and brokers are bilateral, the rate at which a broker meets a firm is $\alpha(b)/b$. Brokers who participate in the market incur a flow cost, κK with $\kappa > 0$. In order to obtain a balanced growth path with a constant α, this entry cost is proportional to the aggregate capital stock.[27]

[25] In the Supplementary Appendix, I set $\beta = 0.75$ and I recalibrate the model. As in Figure 3, the effects of α and η on r and g are nonmonotonic. The static misallocation dominates and the welfare gains from removing frictions, Δ, are always positive.

[26] Pagnotta and Philippon (2018) consider a related environment where trading venues make investments to compete on speed. In the Supplementary Appendix, I endogenize the other dimension of the market structure, the bargaining power of brokers, by allowing them to invest in their bargaining strength.

[27] Alternatively, one could assume that a broker who enters incurs a one-time setup cost. The presence of such fixed cost can give rise to multiple equilibria.

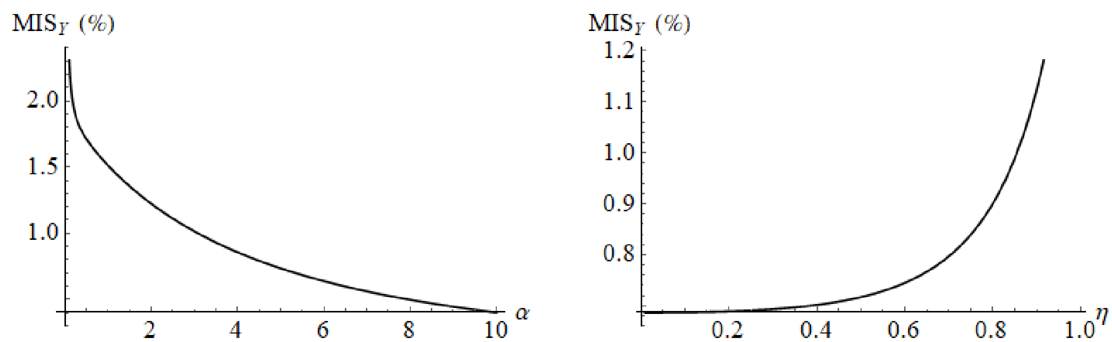

Figure 5: Misallocation measured by the loss of output relative to the first best.

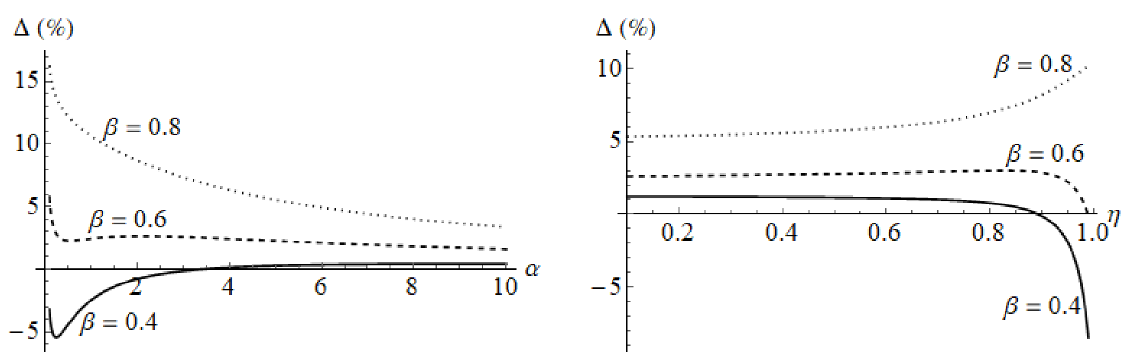

Figure 6: Welfare gain from removing frictions in the market for capital goods.

From (15), the value of a broker who participates in the market solves

$$rw = -\kappa K + \frac{\alpha(b)}{b} \frac{\eta (A_H - A_L) K_L}{r + \lambda + \alpha(b)(1-\eta)} + \dot{w}. \tag{58}$$

The broker meets a firm at rate $\alpha(b)/b$. The firm has low productivity, with probability π_L, and its average capital is K_L/π_L. The gain from reallocating one unit of capital from a low-productivity firm to a high-productivity one is $(A_H - A_L) / [r + \lambda + \alpha(b)(1-\eta)]$. The broker captures a share η of that gain. The last term on the right side of (58) is the increase in the value of the broker over time. Free entry implies $w_t = 0$ for all t. Thus, after substituting $\hat{k}_L$ by its expression, (58) can be rewritten as

$$\frac{b}{\alpha(b)} = \frac{\theta\eta (A_H - A_L) \lambda\pi_L}{[r + \lambda + \alpha(b)(1-\eta)] \{[\lambda + \alpha(b)] \theta + r - \rho\} \kappa}. \tag{59}$$

Equation (59) determines the measure of brokers, b, as a function of r. The left side is increasing in b from 0 to $+\infty$ as b varies over $\mathbb{R}^+$. Assuming $\eta > 0$ and $r > \rho$, the right side decreases from a positive value to 0 as b rises from 0 to $+\infty$ due to two effects. There is a competition effect according to which, for given $\hat{k}_L$, the broker's revenue falls as the measure of competitors increases provided that $\eta < 1$. In addition, $\hat{k}_L$ decreases as b increases. Hence, for all r such that $r - \rho > 0$, there is a unique b solution to (59). Moreover, the right side of (59) is decreasing in r for all $r > \rho$. Hence, b is a decreasing function of r.

From (24), the real interest is determined by

$$r = \frac{[r + \alpha(b)(1-\eta)] A_H + \lambda\bar{A}}{r + \lambda + \alpha(b)(1-\eta)}. \tag{60}$$

Assuming $\eta < 1$, the real interest rate solution to (60) is an increasing function of b. When $b = 0$, $r = r_0$ solution to $r = (rA_H + \lambda\bar{A})/(r + \lambda)$. As b tends to $+\infty$, r approaches A_H. If $\eta = 1$, then $r = r_0$ for all $b \geq 0$. An equilibrium is a pair, $(b, r) \in \mathbb{R}^+ \times (\rho, +\infty)$, solution to (59)-(60) and such that $r < \rho/(1-\theta)$ if $\theta < 1$.

The determination of the equilibrium is represented graphically in Figure 7. The curve labelled FE corresponds to (7). As r increases, the discounted profits of the brokers fall, and hence b falls. The curve labelled R corresponds to (60). As the measure of brokers increases, the misallocation of the capital decreases, and the real interest rate increases.

Proposition 10 ***(Endogenous intermediation in the capital market.)*** *Suppose $\rho \approx 0$, $\theta > 1$, and $\eta \in (0,1)$. There exits a unique balanced-growth path equilibrium and it is such that $r \in (r_0, A_H)$.*

1. *A decrease in the entry cost, κ, raises b, r, and g.*

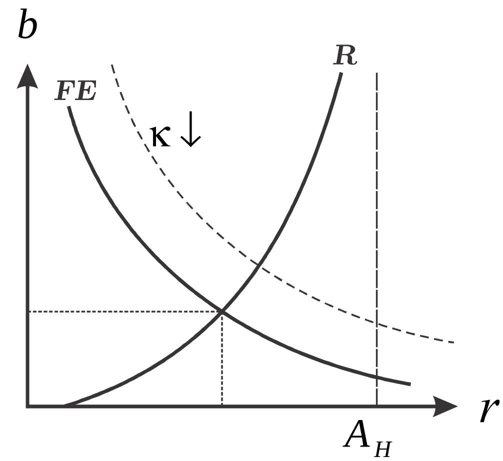

Figure 7: Equilibrium with free-entry of capital market intermediaries.

2. As $\eta \to 0$ or $\eta \to 1$, $r \to r_0$ where $r_0 \in \left(\bar{A}, A_H\right)$. For all $\eta \in (0,1)$, $b > 0$ and $r > r_0$.

The parameter κ can be interpreted as a measure of the barriers to entry. A decrease in κ shifts the FE-curve upward. The measure of brokers, the real interest rate, and the growth rate increase. The second part of Proposition 10 describes the effects of a change in brokers' bargaining power, η, on r and g. As η increases, the FE-curve shifts upward while the R-curve shifts to the left. The effect on r is a priori ambiguous. If $\eta = 0$, brokers do not enter, the capital is not reallocated, and $r = r_0$. If $\eta = 1$, the holdup problem is so severe that firms get no surplus from accessing the capital market. As a result, $r = r_0$. For all values of η between 0 and 1, b is positive and r is greater than r_0. So, r is a nonmonotone function of η that achieves a maximum for some $\eta^* \in (0,1)$.

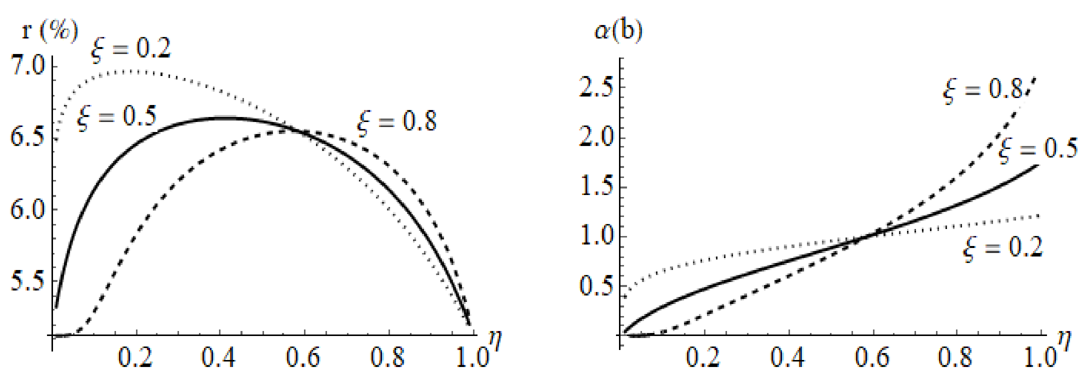

Figure 8: Effects of brokers' bargaining power on the real interest rate and firms' access to the capital market. The matching function is $\alpha(b) = 2b^{\xi}$. The parameters are $\rho = 0.01$, $\theta = 2$, $\kappa = 0.01$, $\lambda = 2$, $A_H = 0.1$, $A_L = 0$, $\pi_H = 0.5$.

Figure 8 provides a numerical example. The matching technology is $\alpha(b) = \alpha_0 b^{\xi}$, where $\xi \in (0, 1)$ is the elasticity of the matching function, which measures the contribution of brokers to the matching process. In the left panel, the relation between r and η is hump-shaped. The real interest rate, and hence the rate of growth of the economy, reaches a maximum for some η^* between 0 and 1. The reason is that brokers play a social role to reallocate capital across firms of different productivity levels, thereby enhancing the rate of return on financial claims. By a logic similar to that underlying the Hosios condition, for a given stock of capital, K_t, brokers' private incentives are aligned with their social role if η coincides with the elasticity of the matching function, ξ. Thus, the greater ξ, the greater η^*. However, because η generates a holdup problem that distorts the real interest rate, η^* is less than ξ.

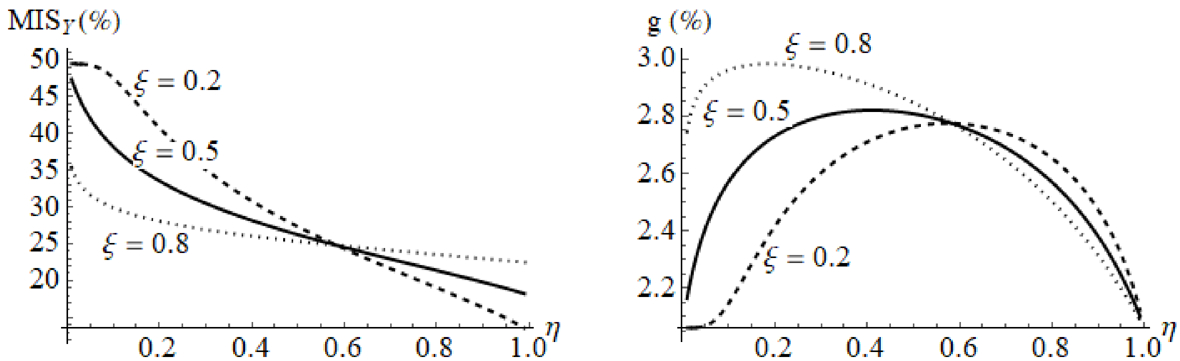

Figure 9: Misallocation and growth under broker entry

In Figure 9, I report the measure of misallocation given by (28) and the growth rate of the economy as a function of η. From the left panel, misallocation decreases with η due to the positive effect on firms' access to the capital market, $\alpha(b)$. In the right panel, there is hump-shape relation between g and η. Therefore, if η is close to one, a reduction in brokers' bargaining power can simultaneously increase misallocation and the economy's growth rate. It is another example of the subtle relation between misallocation and growth.

6 Conclusion

This paper aimed at deepening our understanding of the relationship between the microstructure of the capital goods market, capital misallocation, and long-term economic growth. It explored whether the frictions leading to an inefficient distribution of capital among firms invariably hinder long-run growth. Additionally, it examined whether the recent shift toward online platforms in the market could foster economic growth and enhance social welfare.

In an AK model with heterogeneous firms, and in the absence of knowledge or investment externalities, the answers to these questions aligned with intuition. Market frictions reduce both the real interest rate and the economy's growth rate. The same factors that cause capital misallocation—idiosyncratic productivity risk, search frictions, and brokers' bargaining power—also diminish the economy's growth rate. A key insight of the model is to illustrate how different frictions—the search friction and the bargaining friction—play distinct roles in shaping TFP and the real interest rate. The search friction directly affects the share of capital operated by less productive firms and hence TFP. In contrast, the bargaining friction influences directly the rate of return of financial claims and the growth rate of the capital stock but only indirectly capital misallocation and TFP. When search frictions are small but brokers' bargaining power is large, the real interest rate falls below the average productivity of capital, which is close to its first-best value. Thus, my model can explain why there might be little incentive to invest in one country despite its TFP being large.

In the presence of a learning-by-investing externality, as described by Arrow (1962) or Romer (1986), the relationship between market frictions, misallocation, and growth becomes more nuanced. While search and bargaining frictions still worsen the misallocation of the aggregate capital stock across firms with different productivities, their impact on the real interest rate and the economy's growth rate is non-monotonic. Counterintuitively, technological advancements that improve the rate of capital reallocation across firms may not necessarily enhance growth. Similarly, reforms that reduce brokers' or dealers' market power can, under certain conditions, lower the economy's growth rate. Even more strikingly, I demonstrate that the growth rate of an economy with frictional capital markets can exceed that of a Walrasian economy. These results arise because firms' optimal responses to mitigate capital market frictions tend to alleviate the knowledge externality linked to capital accumulation. From a welfare perspective, greater frictions may even correspond to higher lifetime utility for households.

From a methodological perspective, this paper developed a model of endogenous growth featuring heterogeneous firms and frictional capital markets. The model is remarkably tractable, allowing for closed-form solutions for firm valuations, Tobin's q, firms' demand for capital, the distribution of capital across firms, and the equilibrium real interest rate. Possible extensions include incorporating entry by firm, examining alternative pricing mechanisms (e.g., auctions), exploring different descriptions of knowledge externalities, modeling embodied technological progress, or introducing liquidity constraints and financial frictions. Some of these extensions are discussed in the supplementary appendices.

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Proofs of the main propositions and lemmas

Proof of Proposition 2. The equilibrium real interest rate along a balanced-growth path with positive growth is a $r \in (\rho, +\infty)$ solution to (24), i.e.,

$$r = RHS(r) \equiv \frac{[r + \alpha(1-\eta)]\, A_H + \lambda \bar{A}}{r + \lambda + \alpha(1-\eta)}.$$

The derivative of the right side is

$$\frac{\partial RHS}{\partial r} = \frac{\lambda \left(A_H - \bar{A}\right)}{[r + \lambda + \alpha(1-\eta)]^2} > 0.$$

The slope is positive and it is decreasing in r, i.e., $RHS''(r) < 0$. So RHS is an increasing and concave function of r. Moreover,

$$\begin{aligned} RHS(0) &= \frac{\alpha(1-\eta)A_H + \lambda \bar{A}}{\alpha(1-\eta) + \lambda} \in \left(\bar{A}, A_H\right) \\ RHS(+\infty) &= A_H. \end{aligned}$$

Hence, there is a unique $r > 0$ solution to (24) and it belongs to $\left(\bar{A}, A_H\right)$. The solution is such that $r > \rho$ if and only if $RHS(\rho) > \rho$, i.e.,

$$\frac{[\rho + \alpha(1-\eta)]\, A_H + \lambda \bar{A}}{\rho + \lambda + \alpha(1-\eta)} > \rho.$$

This condition can be rewritten as (26).

The lifetime utility of the household along a balanced-growth equilibrium is

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(c_0 e^{gt}\right)^{1-\theta}}{1-\theta} dt = \frac{c_0^{1-\theta}}{(1-\theta)\,[\rho - (1-\theta)g]},$$

is bounded if $\rho - (1-\theta)g > 0$, i.e., $\rho + (\theta - 1)r > 0$, where I used the Euler equation according to which $g = (r-\rho)/\theta$. It is also equivalent to $r - g > 0$. If $\theta \geq 1$, this inequality holds for all $r > 0$. Suppose $\theta < 1$. The condition $r - g > 0$ can be rewritten as $r < \rho/(1-\theta)$. This inequality holds if and only if $RHS\left(\rho/(1-\theta)\right) < \rho/(1-\theta)$, i.e., (27) holds.

In order to obtain the comparative statics, I make use of the graphical determination of r, denoted r^e, in Figure 2. The RHS increases as α increases, η decreases, or λ decreases. Hence, $\partial r/\partial \alpha > 0$, $\partial r/\partial \eta < 0$, and $\partial r/\partial \lambda < 0$. ■

Proof of Lemma 2. I decompose the value function as illustrated in Figure 10:

$$\upsilon_{j,t}(k) = F_{j,t}(k) + \mathbb{E}\left[e^{-r(T-t)} \max_{k' \geq 0} \left\{\upsilon_{j',T}(k') - (k' - k)\right\}\right], \tag{61}$$

where $F_{j,t}(k) \equiv \mathbb{E}\left[\int_t^T e^{-r(s-t)} K_s^{1-\beta} A_{x(s)} k^\beta ds\right]$ is the expected discounted sum of output flows until the next *effective* meeting with a broker at time T, where $T-t$ is exponentially distributed with mean $1/\alpha(1-\eta)$, and $x(s)$ is the idiosyncratic productivity type of the firm at time $s \in [t,T]$ with $x(t)=j$. The second term on the right side of (61) is the expected discounted value of the firm at time T when it has the opportunity to readjust its capital. The firm invests $k'-k$ and its continuation value is $v_{j',T}(k')$ where $j'=x(T) \in \{1,...,J\}$ is the idiosyncratic productivity type at time T. The expectation is taken with respect to (T,j').

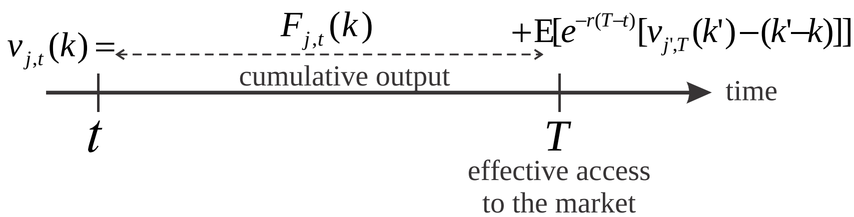

Figure 10: Recursive dermination of the value of a firm

Since aggregate productivity grows at rate $(1-\beta)g$, I detrend cumulative output by defining

$$\begin{aligned}
\tilde{F}_j(k) &\equiv e^{-(1-\beta)gt} F_{j,t}(k) \\
&= e^{-(1-\beta)gt} \mathbb{E}\left[\int_t^T e^{-r(s-t)} e^{(1-\beta)gs} K_0^{1-\beta} A_{x(s)} k^\beta ds\right] \\
&= \mathbb{E}\left[\int_t^T e^{-[r-(1-\beta)g](s-t)} K_0^{1-\beta} A_{x(s)} k^\beta ds\right],
\end{aligned}$$

where to obtain the second equality I used that $K_s = e^{gs} K_0$ and to obtain the third equality I rewrote $e^{(1-\beta)gs} = e^{(1-\beta)g(s-t)} e^{(1-\beta)gt}$. Using the change of variable, $u=s-t$, and the fact that the distribution of $T-t$ is exponential,

$$\tilde{F}_j(k) = \int_0^{+\infty} \alpha(1-\eta) e^{-\alpha(1-\eta)z} \int_0^z e^{-[r-(1-\beta)g]u} K_0^{1-\beta} A_{x(t+u)} k^\beta du dz.$$

By changing the order of integration, it can be rewritten as follows:

$$\tilde{F}_j(k) = \int_0^{+\infty} e^{-[r-(1-\beta)g+\alpha(1-\eta)]u} K_0^{1-\beta} A_{x(t+u)} k^\beta du.$$

Hence, $\tilde{F}_j(k)$ only depends on t through the initial idiosyncratic productivity state, $x(t)=j$. It can be seen from this expression that $\tilde{F}_j(k)$ can be interpreted as the discounted of the output flows of the firm, $A_j K_0^{1-\beta} k^\beta$, over an infinite time horizon where the effective discount rate is $r - (1-\beta)g + \alpha(1-\eta)$. It can then be written recursively as follows:

$$[r - (1-\beta)g + \alpha(1-\eta)] \tilde{F}_j(k) = A_j K_0^{1-\beta} k^\beta + \lambda \sum_{x=1}^{J} \pi_x \left[\tilde{F}_x(k) - \tilde{F}_j(k)\right]. \tag{62}$$

Multiply by π_j both sides and sum across j's to obtain:

$$\sum_{j=1}^{J} \pi_j \tilde{F}_j(k) = \frac{\bar{A} K_0^{1-\beta} k^\beta}{r - (1-\beta)g + \alpha(1-\eta)}.$$

Substitute this expression back into (62) to obtain:

$$\tilde{F}_j(k) = \left\{ \frac{[r - (1-\beta)g + \alpha(1-\eta)] A_j + \lambda \bar{A}}{[r - (1-\beta)g + \alpha(1-\eta) + \lambda] [r - (1-\beta)g + \alpha(1-\eta)]} \right\} K_0^{1-\beta} k^\beta.$$

Using the definition for $\hat{A}_j$ in (35), i.e.,

$$\hat{A}_j \equiv \frac{[r - (1-\beta)g + \alpha(1-\eta)] A_j + \lambda \bar{A}}{r - (1-\beta)g + \alpha(1-\eta) + \lambda},$$

the solution can be rewritten as

$$\tilde{F}_j(k) = \frac{\hat{A}_j K_0^{1-\beta} k^\beta}{r - (1-\beta)g + \alpha(1-\eta)}. \tag{63}$$

The first term on the right side of (34) is equal to $F_{j,t}(k) = e^{(1-\beta)gt} \tilde{F}_j(k)$ where $\tilde{F}_j(k)$ is given by (63). The second and third terms correspond to

$$\mathbb{E}\left[e^{-r(T-t)} \max_{k' \geq 0} \{v_{j',T}(k') - (k' - k)\}\right] = \mathbb{E}\left[e^{-r(T-t)}\right] k + \mathbb{E}\left[e^{-r(T-t)} \max_{k' \geq 0} \{v_{j',T}(k') - k'\}\right].$$

The first term on the right side is the discounted value of the capital that can be liquidated at price one at time T. Using that $T - t$ is exponentially distributed with mean $1/\alpha(1-\eta)$,

$$\mathbb{E}\left[e^{-r(T-t)}\right] = \int_0^{+\infty} \alpha(1-\eta) e^{-\alpha(1-\eta)s} e^{-rs} ds = \frac{\alpha(1-\eta)}{r + \alpha(1-\eta)}.$$

Hence, $\mathbb{E}\left[e^{-r(T-t)}\right] k = \alpha(1-\eta)k / [r + \alpha(1-\eta)]$. The second term on the right side of the equation above,

$$\Lambda_{j,t} \equiv \mathbb{E}\left[e^{-r(T-t)} \max_{k' \geq 0} \{v_{j',T}(k') - k'\}\right],$$

corresponds to the firm re-optimizing its value by choosing k'. It is independent of k. ■

Proof of Proposition 5. By substituting $g = (r - \rho)/\theta$ into (42) one obtains

$$r = RHS(r), \tag{64}$$

where

$$RHS(r) \equiv \frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)}\left\{\frac{\alpha\theta}{\alpha\theta+r-\rho}\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta}$$

and where, from (35), with a slight abuse of notation,

$$\hat{A}_j(r) \equiv \frac{\left\{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)\right\}A_j+\theta\lambda\bar{A}}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)+\theta\lambda}. \tag{65}$$

In order to establish that a solution exists and is such that $r>\min\{\rho,g\}$, I distinguish three cases.

Case #1: $\theta\geq 1$. The term $\theta-(1-\beta)$ is positive. For all $r\geq\rho$, $RHS(r)$ is a continuous function, where I used that $\hat{A}_j(r)$ is continuous for all $r>0$. As $r\to+\infty$,

$$\begin{aligned}\frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)} &\to \frac{\theta\beta}{\theta-(1-\beta)}\in(0,+\infty)\\ \frac{\alpha\theta}{\alpha\theta+r-\rho}\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}} &\to 0,\end{aligned}$$

where I used that

$$\lim_{r\to+\infty}\hat{A}_j(r)=A_j.$$

Hence, $\lim_{r\to+\infty}RHS(r)=0$. A sufficient condition for the existence of a solution to $r=RHS(r)$ such that $r>\rho$ is $RHS(\rho)>\rho$, i.e.,

$$\beta\left\{\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(\rho)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta}>\rho.$$

By replacing $\hat{A}_j(\rho)$ by its expression given by (65), i.e.,

$$\hat{A}_j(\rho)\equiv\frac{\left[\rho+\alpha(1-\eta)\right]A_j+\lambda\bar{A}}{\rho+\alpha(1-\eta)+\lambda},$$

one obtains (43). The condition $r>g=(r-\rho)/\theta$, i.e., $(\theta-1)r+\rho>0$, is automatically satisfied when $\theta\geq 1$.

Case #2: $1-\beta\leq\theta<1$. Since $\theta-(1-\beta)\geq 0$, $RHS(r)$ is continuous in r for all $r\geq\rho$. As in Case #1, a sufficient condition for $r>\rho$ is $RHS(\rho)>\rho$, i.e., (43) holds. The condition, $r>g$, can be rewritten as $r<\bar{r}_\theta\equiv\rho/(1-\theta)$. A sufficient condition for $r<\bar{r}_\theta$ is $RHS(\bar{r}_\theta)<\bar{r}_\theta$, i.e., (44) holds.

Case #3: $\theta<1-\beta$. Given that $\theta-(1-\beta)<0$, the denominator of $RHS(r)$ approaches 0^+ as r approaches $r_0\equiv\left[(1-\beta)\rho+\theta\alpha(1-\eta)\right]/\left[1-\beta-\theta\right]>0$ by below. It can be checked that $r_0 > \bar{r}_\theta$, i.e.,

$$\frac{(1-\beta)\rho + \theta\alpha(1-\eta)}{1-\beta-\theta} > \frac{\rho}{1-\theta} \Longleftrightarrow (1-\beta)\rho\,(1-\theta) + \theta\alpha(1-\eta)\,(1-\theta) > \rho\,(1-\beta-\theta)$$
$$\Longleftrightarrow \beta\rho + \alpha(1-\eta)\,(1-\theta) > 0,$$

where I used that $1-\theta > \beta > 0$. Hence, as in Case #2, the conditions $RHS(\rho) > \rho$ and $RHS(\bar{r}_\theta) < \bar{r}_\theta$, i.e., (43) and (44), guarantee that there exists a $r \in (\rho, \bar{r}_\theta)$.

Next, I establish that any solution to (64) that is such that $r > \min\{\rho, g\}$ is bounded above by r^* defined by (31). Consider the first term on the right side of (64),

$$\Psi(\beta; r) \equiv \frac{\theta\beta\,[\alpha(1-\eta) + r]}{r\,[\theta - (1-\beta)] + (1-\beta)\rho + \theta\alpha(1-\eta)}.$$

The derivative with respect to β is

$$\Psi'(\beta; r) \equiv \frac{\theta^2\,[\alpha(1-\eta) + r]\,[r - g + \alpha(1-\eta)]}{\{r\,[\theta - (1-\beta)] + (1-\beta)\rho + \theta\alpha(1-\eta)\}^2} > 0 \text{ for all } r > g.$$

Moreover, $\lim_{\beta \to 1} \Psi(\beta) = 1$. Hence, for all $r > \min\{\rho, g\}$, and for all $\beta \leq 1$, $\Psi(\beta) \leq 1$. Thus, from (64),

$$r \leq \left\{ \sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}} \right\}^{1-\beta}.$$

From (65), the transformation from $\{A_j\}_{j=1}^J$ to $\{\hat{A}_j\}_{j=1}^J$ is a mean-preserving decrease in spread. Since $x^{\frac{1}{1-\beta}}$ is a convex function for all $\beta < 1$,

$$\sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}} \leq \sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}.$$

It follows that $r \leq r^*$. ■

APPENDIX A.
PROOFS OF OTHER LEMMAS AND PROPOSITIONS

Proof of Proposition 3. The planner's problem is:

$$\max_{C_t,\mu_t,K_{H,t},K_{L,t}} \int_0^{+\infty} e^{-\rho t} u(C_t) dt \tag{66}$$

$$\begin{aligned} \text{s.t. } \dot{K}_{H,t} &= \mu_t \Upsilon\left(C, K_{H,t}, K_{L,t}\right) + \lambda \pi_H K_{L,t} - \left(\lambda \pi_L + \alpha\right) K_{H,t} && (67) \\ \dot{K}_{L,t} &= (1-\mu_t)\Upsilon\left(C, K_{H,t}, K_{L,t}\right) + \lambda \pi_L K_{H,t} - \left(\lambda \pi_H + \alpha\right) K_{L,t}, && (68) \end{aligned}$$

where

$$\Upsilon\left(C, K_H, K_L\right) \equiv A_H K_H + A_L K_L - C + \alpha(K_H + K_L).$$

is the flow of goods that can be allocated in the form of capital to type-H and type-L firms. It has two components. First, there is the new capital, which is equal to total output net of consumption, $A_H K_H + A_L K_L - C$. Second, there is the existing capital that becomes available to be reallocated, $\alpha(K_H + K_L)$. According to (66), the planner maximizes the lifetime discounted utility of the representative household by choosing the time-paths for consumption, C_t, the share of Υ that is allocated to high-productivity firms, μ_t, and the aggregate capital stocks of high and low productivity firms, $K_{H,t}$ and $K_{L,t}$. The terms $\lambda \pi_H K_L$ and $\lambda \pi_L K_H$ in (67) and (68) correspond to the capital owned by firms that are subject to idiosyncratic productivity shocks. The terms αK_H and αK_L correspond to the flows of capital that can be reallocated through brokers.

Necessary conditions. I apply the Maximum Principle to obtain the necessary conditions for an optimum. The current-value Hamiltonian is $\mathcal{H} \equiv u(C) + \xi_H \dot{K}_H + \xi_L \dot{K}_L$, where ξ_H and ξ_L denote the co-state variables. The optimal consumption satisfies

$$u'(C_t) = \mu_t \xi_{H,t} + (1-\mu_t)\xi_{L,t} \quad \forall t \geq 0.$$

The first-order condition for μ_t is

$$\mu_t \begin{matrix} = 1 \\ \in [0,1] \\ = 0 \end{matrix} \quad \text{if} \quad \Upsilon\left(C_t, K_{H,t}, K_{L,t}\right)\left(\xi_{H,t} - \xi_{L,t}\right) \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \forall t \geq 0.$$

The co-state variables solve

$$\begin{aligned} \rho \xi_{H,t} &= \xi_{H,t}\left[\mu_t(A_H + \alpha) - (\lambda \pi_L + \alpha)\right] + \xi_{L,t}\left[(1-\mu_t)(A_H+\alpha) + \lambda \pi_L\right] + \dot{\xi}_{H,t} \\ \rho \xi_{L,t} &= \xi_{H,t}\left[\mu_t\left(A_L + \alpha\right) + \lambda \pi_H\right] + \xi_{L,t}\left[(1-\mu)(A_L+\alpha) - (\lambda \pi_H + \alpha)\right] + \dot{\xi}_{L,t}, \end{aligned}$$

for all $t \geq 0$.

Conjecture. I conjecture the solution is such that $\dot{K}_t = A_H K_{H,t} + A_L K_{L,t} - C_t > 0$ and $\xi_{H,t} > \xi_{L,t}$ for all t. Hence, from the first-order condition above, $\mu_t = 1$, i.e., all the capital that is available for (re)allocation, Υ, is allocated to high-productivity firms. Under this conjecture, the ODEs for $\xi_{H,t}$ and $\xi_{L,t}$ are given by:

$$\rho\xi_{H,t} = \xi_{H,t}A_H + \lambda\pi_L(\xi_{L,t} - \xi_{H,t}) + \dot{\xi}_{H,t} \quad (69)$$

$$\rho\xi_{L,t} = \xi_{H,t}A_L + (\lambda\pi_H + \alpha)(\xi_{H,t} - \xi_{L,t}) + \dot{\xi}_{L,t}, \quad (70)$$

for all $t \geq 0$. The co-state, ξ_H, is the social value of a unit of capital invested in high-productivity firms and it is also the marginal utility of consumption. From the right side of (69), a unit of capital in type-H firms yields A_H units of consumption valued at ξ_H. At Poisson arrival rate $\lambda\pi_L$, the firm's productivity becomes low and the social value of capital switches from ξ_H to ξ_L. Equation (70) has a similar interpretation with an additional term, $\alpha(\xi_H - \xi_L)$, corresponding to the social capital gain when the unit of capital held by a low-productivity firm can be reallocated to a high-productivity firm at Poisson arrival rate α.

I restrict my attention to solutions such that $\dot{\xi}_H/\xi_H = \dot{\xi}_L/\xi_L = \nu$ is constant. From the first-order condition for C, $u'(C_t) = \xi_{H,t}$, it follows that $\dot{C}_t/C_t = g^* \equiv -\nu/\theta$ for all $t \geq 0$. I conjecture a balance-growth path solution such that $\dot{K}_{H,t}/K_{H,t} = \dot{K}_{L,t}/K_{L,t} = g^*$. From (67)-(68),

$$C_t = A_H K_{H,t} + A_L K_{L,t} + \alpha K_t + \lambda\pi_H K_{L,t} - (\lambda\pi_L + \alpha + g^*) K_{H,t} \quad (71)$$

$$\frac{K_{L,t}}{K_t} = \frac{\lambda\pi_L}{\lambda + \alpha + g^*}, \quad (72)$$

for all $t \geq 0$. So, (71) determines C_0 given $(K_{H,0}, K_{L,0})$. According to (72) a balanced-growth path necessitates that the initial share of the capital allocated to low-productivity firms is $\lambda\pi_L/(\lambda + \alpha + g^*)$. Combining (71) and (72),

$$C_0 = A_H K_{H,0} + A_L K_{L,0} - g^* K_0.$$

Denote $\varrho \equiv \rho - \nu = \rho + \theta g^*$, which is the analog of r in the decentralized equilibrium. One can rewrite (69) and (70) as follows:

$$\varrho\xi_{H,t} = \xi_{H,t}A_H + \lambda\pi_L(\xi_{L,t} - \xi_{H,t}) \quad (73)$$

$$\varrho\xi_{L,t} = \xi_{H,t}A_L + (\lambda\pi_H + \alpha)(\xi_{H,t} - \xi_{L,t}). \quad (74)$$

Solving (73)-(74) with respect to ϱ and ξ_L/ξ_H, I obtain:

$$\varrho = \frac{(\varrho + \lambda\pi_H + \alpha) A_H + \lambda\pi_L A_L}{\varrho + \lambda + \alpha} \tag{75}$$

$$\frac{\xi_{L,t}}{\xi_{H,t}} = \frac{A_L + \lambda\pi_H + \alpha}{\varrho + \lambda\pi_H + \alpha}. \tag{76}$$

There is a unique $\varrho > 0$ solution to (75). Finally, using that $\xi_{H,0} = u'(C_0)$, (76) determines $\xi_{L,0}$ as follows:

$$\xi_{L,0} = \frac{A_L + \lambda\pi_H + \alpha}{\varrho + \lambda\pi_H + \alpha} u'(C_0).$$

Comparison to decentralized equilibrium. I compare ϱ given by (75) to r given by (24). They coincide if and only if $\eta = 0$. It follows that $g^* = g$ if and only if $\eta = 0$. Since C_0 is determined as in the decentralized equilibrium, if $\eta = 0$, all real variables, C_t, $K_{L,t}$, and $K_{H,t}$, coincide with those in equilibrium. Similarly, the shadow price, ξ_L/ξ_H, given by (76) coincides with q_L given by (11),

$$q_L = \frac{A_L + \lambda\pi_H + \alpha(1-\eta)}{r + \lambda\pi_H + \alpha(1-\eta)},$$

if and only if $\eta = 0$.

Sufficiency condition. I now check the Arrow sufficiency condition for optimality. The Hamiltonian maximized with respect to μ and C is

$$\begin{aligned} \hat{H}(K_H, K_L, \xi_H, \xi_L) \equiv & \max_{C \geq 0} \{u(C) - \xi_H C\} \\ & + \xi_H \left[A_H K_H + A_L K_L + \alpha(K_H + K_L) + \lambda\pi_H K_L - (\lambda\pi_L + \alpha) K_H\right] \\ & + \xi_L \left[\lambda\pi_L K_H - (\lambda\pi_H + \alpha) K_L\right]. \end{aligned}$$

It is linear in K_H and K_L and, hence, it is concave in (K_H, K_L). Moreover,

$$\lim_{t\to+\infty} e^{-\rho t}\xi_{H,t}K_{H,t} = \lim_{t\to+\infty} e^{-\left[\frac{\rho-(1-\theta)\varrho}{\theta}\right]t}\xi_{H,0}K_{H,0}$$

where I used that

$$\dot{K}_H/K_H = \dot{C}/C = -\nu/\theta.$$

By assumption, $\rho > A_H(1-\theta)$. Hence, $\rho - (1-\theta)\varrho > 0$ and $\lim_{t\to+\infty} e^{-\rho t}\xi_{H,t}K_{H,t} = 0$. By a similar reasoning, $\lim_{t\to+\infty} e^{-\rho t}\xi_{L,t}K_{L,t} = 0$. So the proposed solution is an optimum. ■

Proof of Proposition 4. Part 1: $\eta < 1$. From Proposition 2, r is bounded above by A_H. The result that $\lim_{\alpha\to+\infty} r = A_H$ follows directly from (24), i.e.,

$$\lim_{\alpha\to+\infty} r = \lim_{\alpha\to+\infty} \left\{A_H - \frac{\lambda\pi_L(A_H - A_L)}{r + \lambda + \alpha(1-\eta)}\right\} = A_H.$$

Given r, g is determined by (5). The share of capital allocated to high productivity firms, $\hat{k}_H$, is given by (17). It approaches one as $\alpha \to +\infty$. From (8),

$$\phi_j(k) = \eta\,(q_j - 1)\max_{k' \geq 0}(k' - k), \quad j \in \{L, H\}.$$

Since $p_K = q_H = 1$, intermediation fees for high-productivity firms are always 0, $\phi_H(k) = 0$ for all k. Intermediation fees for low-productivity firms are given by (14). Thus,

$$\lim_{\alpha \to +\infty} \phi_L(k) = \lim_{\alpha \to +\infty} \frac{\eta\,(A_H - A_L)\,k}{r + \lambda + \alpha(1 - \eta)} = 0,$$

provided that $\eta < 1$.

Part 2: $\eta = 1$. From (24), if $\eta = 1$, r is a solution to the following quadratic equation,

$$r^2 + (\lambda - A_H)r - \lambda\bar{A} = 0.$$

The discriminant is positive, so the equation admits two roots. The product of the roots is negative, so the roots have opposite sign. I focus on the positive root since $r > 0$ is a necessary condition for an equilibrium with positive growth. When evaluated at $r = A_H$, the left side is equal to $\lambda(A_H - \bar{A}) > 0$. Hence, $r < A_H$. In closed form, the positive solution is

$$r = \frac{A_H - \lambda + \sqrt{(A_H - \lambda)^2 + 4\lambda\bar{A}}}{2} < A_H.$$

The growth rate, g, and the share of capital allocated to high productivity firms, $\hat{k}_H$, are determined as in Part 1. Finally, the intermediation fees for low-productivity firms are determined by (14),

$$\phi_L(k) = \frac{(A_H - A_L)\,k}{r + \lambda}.$$

For all $k > 0$, $\phi_L(k)$ is independent of α and bounded away from 0. ■

Proof of Lemma 1.

The planner's problem is:

$$\max_{\{c_t, k_{j,t}, K_t\}} \int_0^{+\infty} e^{-\rho t}\frac{(c_t)^{1-\theta}}{1-\theta}dt \text{ s.t. } c_t = Y_t - \dot{K}_t$$

$$\text{where } Y_t = \sum_{j=1}^{J} \pi_j A_j K_t^{1-\beta} k_{j,t}^{\beta} \text{ and } K_t = \sum_{j=1}^{J} \pi_j k_{j,t}.$$

The planner maximizes the lifetime utility of the representative household subject to law of motion of K_t, the definition of aggregate output as the sum of the output flows of a unit measure of firms divided into J types, and the definition of the aggregate capital stock. The variable, $k_{j,t}$, denotes the capital allocated to type-j firms. In the absence of search frictions, it is a control variable. The current-value Hamiltonian is

$$\mathcal{H}(c, K, k_j, \xi, \mu) \equiv \frac{c^{1-\theta}}{1-\theta} + \xi \left(\sum_{j=1}^{J} \pi_j A_j K^{1-\beta} k_j^{\beta} - c \right) + \mu \left(K - \sum_{j=1}^{J} \pi_j k_j \right),$$

where ξ is the costate variable associated with the law of motion for K_t and μ is the Lagrange multiplier associated with the definition of K_t. By the Maximum Principle,

$$(c_t, \{k_{j,t}\}) \in \arg\max \mathcal{H}(c_t, K_t, k_{j,t}, \xi_t, \mu_t) \text{ for all } t.$$

The first-order conditions with respect to c_t and $k_{j,t}$ are:

$$\begin{aligned} (c_t)^{-\theta} &= \xi_t, \forall t \in \mathbb{R}_+ \\ \beta A_j \left(\frac{K_t}{k_{j,t}} \right)^{1-\beta} &= \frac{\mu_t}{\xi_t} \ \forall j \in \{1, ..., J\}, \forall t \in \mathbb{R}_+. \end{aligned}$$

The second first-order condition can be rexpressed as

$$\frac{k_{j,t}}{K_t} = \left(\beta \frac{\xi_t}{\mu_t} A_j \right)^{\frac{1}{1-\beta}} \quad \forall j \in \{1, ..., J\}, \ \forall t \in \mathbb{R}_+.$$

Multiply both sides by π_j and sum across all j's to obtain:

$$1 = \left(\beta \frac{\xi_t}{\mu_t} \right)^{\frac{1}{1-\beta}} \sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \quad \forall t \in \mathbb{R}_+.$$

Solve for the ratio μ_t/ξ_t:

$$\frac{\mu_t}{\xi_t} = \beta \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta} \quad \forall t \in \mathbb{R}_+.$$

Substitute μ_t/ξ_t into the expression for $k_{j,t}/K_t$ to obtain the share of the capital allocated to firms of type j:

$$\frac{\pi_j k_{j,t}}{K_t} = \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}} \quad \forall j \in \{1, ..., J\}, \ \forall t \in \mathbb{R}_+.$$

It follows that aggregate output is equal to

$$Y_t = K_t \sum_{j=1}^{J} \pi_j A_j \left(\frac{k_{j,t}}{K_t} \right)^{\beta} = K_t \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta} \quad \forall t \in \mathbb{R}_+.$$

From the Maximum Principle, the law of motion of the co-state variable, ξ_t, are given by:

$$\rho\xi_t = \xi_t r^* + \dot{\xi}_t,$$

where I define

$$\begin{aligned} r^* &\equiv \frac{1}{\xi_t}\frac{\partial\mathcal{H}(c_t, K_t, k_{j,t}, \xi_t, \mu_t)}{\partial K_t} \\ &= (1-\beta)\sum_{j=1}^{J}\pi_j A_j\left(\frac{k_{j,t}}{K_t}\right)^{\beta} + \frac{\mu_t}{\xi_t} \\ &= \left(\sum_{j=1}^{J}\pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}. \end{aligned}$$

The third equality was obtained by substituting $k_{j,t}/K_t$ and μ_t/ξ_t by their expressions. It follows that

$$\frac{\dot{\xi}_t}{\xi_t} = \rho - r^* \quad \forall t \in \mathbb{R}_+.$$

Using the first-order condition, $(c_t)^{-\theta} = \xi_t$, it follows that $-\theta\dot{c}_t/c_t = \dot{\xi}_t/\xi_t$. Hence, the growth rate of consumption is

$$g^* = \frac{\dot{c}_t}{c_t} = \frac{r^* - \rho}{\theta}.$$

In order for $g^* > 0$, $\rho < r^*$, which holds from (30). I consider the solution to the necessary conditions such that c_t/K_t is constant. Hence, $\dot{K}_t/K_t = g^*$ and the initial consumption is $c_0 = Y_0 - \dot{K}_0 = Y_0 - g^* K_0$. Using that $Y_t = r^* K_t$,

$$\frac{c_0}{K_0} = r^* - g^*.$$

It is positive if

$$\rho > (1-\theta)\, r^*,$$

which holds from (30).

For sufficiency, I check the Mangasarian condition according to which $\mathcal{H}$ is concave in (c, K, k_j) and

$$\lim_{t\to+\infty} e^{-\rho t}\xi_t K_t = 0.$$

Using that $\dot{\xi}_t/\xi_t = \rho - r^*$,

$$e^{-\rho t}\xi_t K_t = \xi_0 K_0 e^{(g^* - r^*)t}.$$

Since $r^* - g^* > 0$, $\lim_{t\to+\infty} e^{(g^*-r^*)t} = 0$, and the Mangasarian condition holds. ■

Proof of Lemma 3. The density measure of firms of type (i, j, τ) can be written as:

$$\begin{aligned}\gamma(i,j,\tau) &\equiv \Pr\left[(i,j,\tau)\right] \\ &= \Pr\left[j\,|(i,\tau)\right]\times\Pr\left[(i,\tau)\right],\end{aligned}$$

by the definition of conditional expectation. The processes for the productivity type, i, and for the access to the market, τ, are independent. Hence, $\Pr[(i,\tau)] = \Pr[i]\Pr[\tau]$. The long-run probability of type i is $\Pr[i] = \pi_i$. The density of τ is exponential with mean $1/\alpha$, i.e., $\Pr[\tau] = \alpha e^{-\alpha\tau}$. Hence,

$$\Pr\left[(i,\tau)\right] = \pi_i \alpha e^{-\alpha\tau}.$$

Let N_τ denote the number of productivity shocks received by a firm over a time interval of length τ. Its distribution is Poisson with parameter λ. Hence:

$$\begin{aligned}\Pr\left[j\,|(i,\tau)\right] &= \Pr\left[N_\tau = 0\right]\mathbb{I}_{\{i=j\}} + \Pr\left[N_\tau \geq 1\right]\Pr[j] \\ &= e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j.\end{aligned}$$

If $i = j$, the firm can end up with productivity j by not receiving any shock with probability $e^{-\lambda\tau}$. If the firm receives at least one shock, with probability $1 - e^{-\lambda\tau}$, the last realization must be j with probability π_j. By multiplying the expressions for $\Pr[(i,\tau)]$ and $\Pr[j\,|(i,\tau)]$, I obtain (38). ■

Proof of Lemma 4. Along a balanced growth path, $k_{i,t-\tau} = e^{-g\tau}k_{i,t}$. Substituting $\gamma(i,j,\tau)$ by its expression given by (38) into (39),

$$K_{j,t} = \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t} \int_{\mathbb{R}^+} \alpha e^{-(\alpha+g)\tau}\left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right]d\tau.$$

The integral on the right side can be rewritten as:

$$\int_{\mathbb{R}^+} \alpha e^{-(\alpha+g)\tau}\left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right]d\tau = \frac{\alpha}{\alpha+g+\lambda}(\mathbb{I}_{\{i=j\}} - \pi_j) + \frac{\alpha}{\alpha+g}\pi_j.$$

Plug it back into the equation for $K_{j,t}$ to obtain:

$$\begin{aligned}K_{j,t} &= \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t}\left[\frac{\alpha}{\alpha+g+\lambda}(\mathbb{I}_{\{i=j\}} - \pi_j) + \frac{\alpha}{\alpha+g}\pi_j\right] \\ &= \frac{\alpha}{\alpha+g+\lambda}\sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t}\mathbb{I}_{\{i=j\}} + \left(\frac{\alpha}{\alpha+g} - \frac{\alpha}{\alpha+g+\lambda}\right)\pi_j \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t} \\ &= \frac{\alpha}{\alpha+g+\lambda}\pi_j k_{j,t} + \frac{\alpha\lambda}{(\alpha+g)(\alpha+g+\lambda)}\pi_j \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t}.\end{aligned}$$

From market clearing, (41),

$$\alpha \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t} = (\alpha+g)K_t.$$

Hence,

$$K_{j,t} = \frac{\alpha}{\alpha+g+\lambda}\pi_j k_{j,t} + \frac{\lambda \pi_j}{\alpha+g+\lambda}K_t.$$

Divide both sides by K_t:

$$\frac{K_{j,t}}{K_t} = \pi_j \left[\frac{\alpha\left(k_j/K\right)+\lambda}{g+\alpha+\lambda}\right].$$

Substitute (k_j/K) by its expression given by (37) to obtain (40). ■

Proof of Proposition 6. Part 1. Rewrite (64) as

$$r = \frac{\beta\left[\alpha(1-\eta)+r\right]\hat{A}_J}{r-(1-\beta)g+\alpha(1-\eta)}\left[\left(\frac{\alpha}{\alpha+g}\right)\sum_{j=1}^{J}\pi_j\left(\frac{\hat{A}_j}{\hat{A}_J}\right)^{\frac{1}{1-\beta}}\right]^{1-\beta}.$$

For all $j<J$, $\hat{A}_j/\hat{A}_J<1$ and

$$\lim_{\beta\to 1}\left(\frac{\hat{A}_j}{\hat{A}_J}\right)^{\frac{1}{1-\beta}} = \lim_{\beta\to 1} e^{\frac{1}{1-\beta}\ln\left(\frac{\hat{A}_j}{\hat{A}_J}\right)} = 0.$$

Hence,

$$\lim_{\beta\to 1}\left[\left(\frac{\alpha}{\alpha+g}\right)\sum_{j=1}^{J}\pi_j\left(\frac{\hat{A}_j}{\hat{A}_J}\right)^{\frac{1}{1-\beta}}\right]^{1-\beta} = \lim_{\beta\to 1}\left[\frac{\alpha\pi_J}{\alpha+g}\right]^{1-\beta} = 0,$$

provided $\pi_J<1$ and $g\geq 0$ (which is satisfied under the conditions of Proposition 5). It follows that

$$\lim_{\beta\to 1} r = \lim_{\beta\to 1}\frac{\beta\left[\alpha(1-\eta)+r\right]\hat{A}_J}{r-(1-\beta)g+\alpha(1-\eta)} = \lim_{\beta\to 1}\hat{A}_J,$$

where, from (35),

$$\lim_{\beta\to 1}\hat{A}_j = \frac{\left[r+\alpha(1-\eta)\right]A_j+\lambda\bar{A}}{r+\alpha(1-\eta)+\lambda}.$$

I now turn to the distribution of capital across firms. From (40),

$$\begin{aligned}\lim_{\beta\to 1}\frac{K_j}{K} &= \lim_{\beta\to 1}\frac{\pi_j}{g+\alpha+\lambda}\left[\alpha\left\{\frac{\beta\left[\alpha(1-\eta)+r\right]\hat{A}_j}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]}\right\}^{\frac{1}{1-\beta}}+\lambda\right]\\ &= \lim_{\beta\to 1}\frac{\pi_j}{g+\alpha+\lambda}\left[\alpha\left(\frac{\hat{A}_j}{\hat{A}_J}\right)^{\frac{1}{1-\beta}}+\lambda\right],\end{aligned}$$

where I used that $\lim_{\beta\to 1} r = \lim_{\beta\to 1} \hat{A}_J$. For all $j < J$,

$$\lim_{\beta\to 1}\left(\frac{\hat{A}_j}{A_J}\right)^{\frac{1}{1-\beta}} = 0.$$

Hence, $\lim_{\beta\to 1}(K_j/K) = \lambda\pi_j/(g+\alpha+\lambda)$. If $j = J$,

$$\lim_{\beta\to 1}\left(\frac{K_J}{K}\right) = 1 - \sum_{j=1}^{J-1}\lim_{\beta\to 1}\left(\frac{K_J}{K}\right) = 1 - \frac{\lambda(1-\pi_J)}{g+\alpha+\lambda}.$$

Part 2. If $\eta < 1$, for given $r \geq \rho$, as $\alpha \to +\infty$,

$$\begin{aligned}
\frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)} \quad &\to \quad \beta \\
\left\{\frac{\alpha\theta}{\alpha\theta+r-\rho}\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta} \quad &\to \quad \left\{\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta}.
\end{aligned}$$

Using that $\lim_{\alpha\to+\infty}\hat{A}_j = A_j$,

$$r \to \beta\left(\sum_{j=1}^{J}\pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

From (40) the share of capital allocated to firms of type j tends to

$$\lim_{\alpha\to+\infty}\left(\frac{K_j}{K}\right) = \lim_{\alpha\to+\infty}\left[\frac{\alpha\pi_j}{g+\alpha+\lambda}\left\{\frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]}\right\}^{\frac{1}{1-\beta}}\hat{A}_j^{\frac{1}{1-\beta}} + \frac{\pi_j\lambda}{g+\alpha+\lambda}\right].$$

Using that $\lim_{\alpha\to+\infty}\left[\alpha\pi_j/(g+\alpha+\lambda)\right] = \pi_j$, $\lim_{\alpha\to+\infty}\left[\pi_j\lambda/(g+\alpha+\lambda)\right] = 0$, $\lim_{\alpha\to+\infty}\hat{A}_j = A_j$,

$$\lim_{\alpha\to+\infty}\frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]} = \lim_{\alpha\to+\infty}\frac{\beta}{r} = \frac{1}{\left(\sum_{j=1}^{J}\pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}},$$

it follows that

$$\lim_{\alpha\to+\infty}\left(\frac{K_j}{K}\right) = \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J}\pi_j A_j^{\frac{1}{1-\beta}}}.$$

Part 3: In order to obtain an equilibrium, assume $\theta > 1-\beta$. Set $\eta = 1$ in (64)-(65) and take the limit as $\alpha \to +\infty$ to obtain (49)-(50). From (40), if $\eta = 1$, as $\alpha \to +\infty$,

$$\frac{K_j}{K} \to \pi_j\left[\frac{\beta}{r-(1-\beta)g}\right]^{\frac{1}{1-\beta}}\hat{A}_j^{\frac{1}{1-\beta}}.$$

From the definition of g:

$$r-(1-\beta)g=\frac{[\theta-(1-\beta)]\,r+(1-\beta)\rho}{\theta}.$$

From the expression for r given by (49),

$$[\theta-(1-\beta)]\,r+(1-\beta)\rho=\theta\beta\left(\sum_{j=1}^{J}\pi_j\hat{A}_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

Hence,

$$r-(1-\beta)g=\beta\left(\sum_{j=1}^{J}\pi_j\hat{A}_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

It follows that

$$\frac{K_j}{K}\rightarrow\frac{\pi_j\hat{A}_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J}\pi_j\hat{A}_j^{\frac{1}{1-\beta}}}.$$

Part 4. Suppose $A_j=\bar{A}$ for all $j\in\{1,...,J\}$. From (65), $\hat{A}_j=\bar{A}$ and, from (64),

$$r=\frac{\alpha(1-\eta)+r}{\alpha(1-\eta)+r-(1-\beta)g}\left(\frac{\alpha}{\alpha+g}\right)^{1-\beta}\beta\bar{A}$$

where $g=(r-\rho)/\theta$. ■

Proof of Proposition 7. Suppose $\eta=1$. From (52), the real interest rate solves

$$r=\left(\frac{\alpha\theta}{\alpha\theta+r-\rho}\right)^{1-\beta}\beta\bar{A}+(1-\beta)\frac{(r-\rho)}{\theta}.$$

It can be reexpressed as:

$$r=\frac{\theta\left(\frac{\alpha\theta}{\alpha\theta+r-\rho}\right)^{1-\beta}\beta\bar{A}-(1-\beta)\rho}{\theta-1+\beta}.$$

Assuming $\theta-1+\beta>0$, the right side is decreasing in r for all $r\geq\rho$. For all $\alpha>0$,

$$RHS|_{r=\rho}=\frac{\theta\beta\bar{A}-(1-\beta)\rho}{\theta-1+\beta}.$$

It is greater than ρ if and only if $\beta\bar{A}>\rho$. If

$$r\geq\rho+\alpha\theta\left\{\left[\frac{\theta\beta\bar{A}}{(1-\beta)\rho}\right]^{\frac{1}{1-\beta}}-1\right\},$$

then $RHS \leq 0$. Thus, if $\beta \bar{A} > \rho$, there is a unique solution to the equation above and it is such that $r > \rho$. Moreover, for all $r > \rho$, since the right side is increase in α, the solution r is also increasing in α. Consider the limit as $\alpha \to 0$. I rewrite the equilibrium condition as

$$\left(\frac{\theta - 1 + \beta}{\theta}\right)(r - \rho) = \left(\frac{\theta}{\theta + \frac{r-\rho}{\alpha}}\right)^{1-\beta} \beta \bar{A} - \rho.$$

If $\lim_{\alpha \to 0} r \in (\rho, +\infty)$, then the left side is finite while the right side grows unbounded, which is a contradiction. Since r is bounded above (by Proposition 5), it must be that $\lim_{\alpha \to 0} r = \rho$ with

$$\lim_{\alpha \to 0} \left(\frac{r - \rho}{\alpha}\right) = \theta \left[\left(\frac{\beta \bar{A}}{\rho}\right)^{\frac{1}{1-\beta}} - 1\right].$$

Consider next the limit as $\alpha \to +\infty$. From the equilibrium condition above,

$$r \to r_\infty \equiv \frac{\theta \beta \bar{A} - (1 - \beta)\rho}{\theta - 1 + \beta}.$$

Moreover, $r_\infty > \beta \bar{A}$ if and only if $\beta < 1$ and $\beta \bar{A} > \rho$. Thus, if $\beta \bar{A} > \rho$, there exists a $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$, $r > \beta \bar{A}$. ■

Proof of Corollary 2.

From (78), using that $A_j = \bar{A}$ for all j, the output per unit of capital is

$$TFP = \left(\frac{\alpha + g}{\alpha}\right)^{\beta} \left(\frac{\alpha}{\alpha + \beta g}\right) \bar{A}.$$

As $\alpha \to +\infty$, $TFP \to \bar{A}$. The lifetime utility of the representative household is:

$$\mathcal{U} = \frac{(TFP - g)^{1-\theta}}{(1 - \theta)\left[\rho - (1 - \theta) g\right]} K_0,$$

where I used that

$$c_t = (TFP - g) \times K_0 e^{gt}$$

It can be reexpressed as

$$\mathcal{U} = \frac{-K_0}{(TFP - g)^{(\theta - 1)} (\theta - 1) \left[(\theta - 1) g + \rho\right]}.$$

The derivative of the denominator, D, with respect to g is equal to

$$\frac{\partial D}{\partial g} = (\theta - 1)^2 (TFP - g)^{(\theta - 2)} (TFP - \theta g - \rho).$$

For all $g \in \left(\frac{\beta\bar{A}-\rho}{\theta}, \frac{\bar{A}-\rho}{\theta}\right)$, $\partial D/\partial g > 0$, and hence $\partial \mathcal{U}/\partial g > 0$. Suppose $\eta = 1$. From Proposition 7, $r > \beta\bar{A}$. If η falls below one, from Lemma 6, as $\alpha \to +\infty$, $r = \beta A$. Since $\partial \mathcal{U}/\partial g > 0$, the lifetime utility of the household falls. ■

Proof of Proposition 8. Based on (54), a measure of capital misallocation is

$$MIS_K^j = \hat{\Xi}_j - \Xi_j^* + \frac{\lambda}{g+\alpha+\lambda}\left(\pi_j - \hat{\Xi}_j\right). \tag{77}$$

It is equal to the difference between K_j/K and its first-best value. If $MIS_K^j > 0$, too much capital is allocated to firms of type j whereas if $MIS_K^j < 0$ too little capital is allocated to those firms.

Part 1: I establish first that

$$\frac{\pi_1 \hat{A}_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} > \frac{\pi_1 A_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}}.$$

By the definition of $\hat{A}_1$, (35), $\hat{A}_1 > A_1$ and hence $\pi_1 \hat{A}_1^{\frac{1}{1-\beta}} > \pi_1 A_1^{\frac{1}{1-\beta}}$. Using that the lottery, $\{(\hat{A}_j, \pi_j)\}$, corresponds to a mean-preserving reduction in spread of the lottery $\{(A_j, \pi_j)\}$, and $x^{\frac{1}{1-\beta}}$ is a convex function,

$$\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} < \sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}.$$

This proves the inequality.

Next, from the fact that $\hat{A}_1 < \hat{A}_2 < .... < \hat{A}_J$, $\hat{A}_1^{\frac{1}{1-\beta}} < \sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}$ and hence

$$1 - \frac{\hat{A}_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} > 0.$$

From (54), i.e.,

$$\frac{K_1}{K} = \frac{\pi_1 \hat{A}_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} + \frac{\lambda \pi_1}{g+\alpha+\lambda}\left(1 - \frac{\hat{A}_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}}\right)$$

and the observations above, it follows that

$$\frac{K_1}{K} > \frac{\pi_1 A_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}}.$$

Part 2: I establish that

$$\frac{\pi_j \hat{A}_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} < \frac{\pi_j A_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}}.$$

Divide the numerator and denominator of the left side by $\hat{A}_J^{\frac{1}{1-\beta}}$ and the numerator and denominator of the right side by $A_J^{\frac{1}{1-\beta}}$ to obtain:

$$\frac{\pi_j}{\sum_{j=1}^J \pi_j (\hat{A}_j/\hat{A}_J)^{\frac{1}{1-\beta}}} < \frac{\pi_j}{\sum_{j=1}^J \pi_j (A_j/A_J)^{\frac{1}{1-\beta}}}.$$

From (35),

$$\frac{\hat{A}_j}{\hat{A}_J} = \frac{[r-(1-\beta)g+\alpha(1-\eta)]\,A_j+\lambda\bar{A}}{[r-(1-\beta)g+\alpha(1-\eta)]\,A_J+\lambda\bar{A}}.$$

It can be checked that this ratio is decreasing in $\alpha(1-\eta)$. As $\alpha(1-\eta)\to+\infty$, $\hat{A}_j/\hat{A}_J \to A_j/A_J$. Hence, $\hat{A}_j/\hat{A}_J > A_j/A_J$ and

$$\sum_{j=1}^J \pi_j \left(\frac{\hat{A}_j}{\hat{A}_J}\right)^{\frac{1}{1-\beta}} > \sum_{j=1}^J \pi_j \left(\frac{A_j}{A_J}\right)^{\frac{1}{1-\beta}}.$$

This establishes the inequality above. For any α,

$$\frac{\hat{A}_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} > 1.$$

The inequality

$$\frac{K_J}{K} < \frac{\pi_J A_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}}$$

follows from the observations above and (54). ■

Lemma 5 *The average output per unit of capital in the decentralized equilibrium is equal to*

$$TFP = \left(\frac{\alpha+g}{\alpha}\right)^{\beta}\left(\frac{\alpha}{\alpha+\beta g}\right)\left(\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}\right)^{-\beta}\sum_{j=1}^J \pi_j \tilde{A}_j \hat{A}_j^{\frac{\beta}{1-\beta}} \tag{78}$$

where

$$\tilde{A}_j \equiv \frac{(\alpha+\beta g)\,A_j+\lambda\bar{A}}{\alpha+\beta g+\lambda}.$$

Proof of Lemma 5.

Replace $\gamma(i,j,\tau)$ by its expression given by (38) into (56) to obtain:

$$\begin{aligned} TFP &= \int_0^{+\infty} \sum_{i,j} e^{-\beta g \tau} A_j \tilde{k}_i^{\beta} \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j \right] d\tau \\ &= \int_0^{+\infty} \sum_i \alpha e^{-(\alpha+\beta g)\tau} \left[e^{-\lambda\tau} A_i + (1-e^{-\lambda\tau})\bar{A} \right] \pi_i \tilde{k}_i^{\beta} d\tau \\ &= \sum_i \left[\frac{\alpha}{\alpha+\beta g+\lambda} \left(A_i - \bar{A} \right) + \frac{\alpha}{\alpha+\beta g} \bar{A} \right] \pi_i \tilde{k}_i^{\beta}. \end{aligned}$$

The second equality sums first over all j's and uses that $\sum_j \mathbb{I}_{\{i=j\}} A_j = A_i$ and $\sum_j \pi_j A_j = \bar{A}$. The third equality interchanges the sum and the integral and it integrates over τ. It uses that $\int_0^{+\infty} \alpha e^{-(\alpha+\beta g+\lambda)\tau} d\tau = \alpha/(\alpha+\beta g+\lambda)$ and $\int_0^{+\infty} \alpha e^{-(\alpha+\beta g)\tau} d\tau = \alpha/(\alpha+\beta g)$. Substitute $\tilde{k}_i^{\beta}$ by its expression given by (37), i.e.,

$$\tilde{k}_i^{\beta} = \left\{ \frac{\beta \left[r + \alpha(1-\eta) \right]}{r \left[r + \alpha(1-\eta) - (1-\beta) g \right]} \right\}^{\frac{\beta}{1-\beta}} \hat{A}_i^{\frac{\beta}{1-\beta}},$$

to obtain

$$TFP = \left\{ \frac{\beta \left[\alpha(1-\eta) + r \right]}{r \left[r - (1-\beta) g + \alpha(1-\eta) \right]} \right\}^{\frac{\beta}{1-\beta}} \sum_i \pi_i \left[\frac{\alpha}{\alpha+\beta g+\lambda} \left(A_i - \bar{A} \right) + \frac{\alpha}{\alpha+\beta g} \bar{A} \right] \hat{A}_i^{\frac{\beta}{1-\beta}}.$$

From the definition $\tilde{A}_i \equiv \left[(\alpha+\beta g) A_i + \lambda \bar{A} \right] / (\alpha+\beta g+\lambda)$,

$$TFP = \frac{\alpha}{(\alpha+\beta g)} \left\{ \frac{\beta \left[\alpha(1-\eta) + r \right]}{r \left[r - (1-\beta) g + \alpha(1-\eta) \right]} \right\}^{\frac{\beta}{1-\beta}} \sum_i \pi_i \tilde{A}_i \hat{A}_i^{\frac{\beta}{1-\beta}}.$$

From the definition of r given by (42):

$$\left\{ \frac{\beta \left[\alpha(1-\eta) + r \right]}{r \left[r - (1-\beta) g + \alpha(1-\eta) \right]} \right\}^{\frac{\beta}{1-\beta}} = \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{-\beta} \left(\frac{\alpha+g}{\alpha} \right)^{\beta}.$$

Substitute this expression into the equation for TFP above to obtain (78). ■

Proof of Proposition 9. I first derive (57). In any equilibrium, $g < r < r^*$. Thus, r is bounded above. If α and λ are large, then $\tilde{A}_j \approx \left(\alpha A_j + \lambda \bar{A} \right) / (\alpha+\lambda)$. If in addition $\eta = 0$, from (35), $\hat{A}_j \approx \left(\alpha A_j + \lambda \bar{A} \right) / (\alpha+\lambda)$. Hence, using that $\tilde{A}_j \approx \hat{A}_j$,

$$\left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{-\beta} \sum_{j=1}^{J} \pi_j \tilde{A}_j \hat{A}_j^{\frac{\beta}{1-\beta}} \approx \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{1-\beta}.$$

Moreover, because α is large,

$$\left(\frac{\alpha+g}{\alpha}\right)^{\beta}\left(\frac{\alpha}{\alpha+\beta g}\right)\approx 1.$$

Putting these observations together, (78) can be rewritten as (57). A decrease in α or an increase in λ generates a mean-preserving decrease in the spread of the distribution of $\hat{A}_j$. Since $x^{\frac{1}{1-\beta}}$ is a convex function, it reduces $\sum_{j=1}^{J}\pi_j\hat{A}_j^{\frac{1}{1-\beta}}$. Hence, from (57), it reduces TFP. Thus, MIS_Y increases. ■

Proof of Proposition 10. Existence and uniqueness. Since $\rho\approx 0$, the growth rate, $g=(r-\rho)/\theta$, is positive whenever $r>0$. According to (59), for all $\eta>0$, b is a decreasing function of $r>0$. It is represented by the curve labelled FE in Figure 7. As r goes to 0, b tends to $b_0>0$ solution to

$$\frac{b_0}{\alpha(b_0)}=\frac{\eta\left(A_H-A_L\right)\lambda\pi_L}{\left[\lambda+\alpha(b_0)(1-\eta)\right]\left[\lambda+\alpha(b_0)\right]\kappa}$$

As r goes to $+\infty$, b approaches 0. According to (60), provided $\eta<1$, r is an increasing function of b. It is represented by the curve labelled R in Figure 7. As b goes to $+\infty$, r tends to A_H. As b goes to 0, r approaches $r_0\in\left(\bar{A},A_H\right)$ solution to

$$r_0=\frac{r_0A_H+\lambda\bar{A}}{r_0+\lambda}.$$

Since $\bar{A}>0$, $r_0>0$. Hence, there is a unique pair, $(b,r)\in\mathbb{R}^+\times(r_0,A_H)$, solution to (59) and (60).

Part 1. Effects of the entry cost. As κ decreases, the FE curve shifts upward and both R and b increase.

Part 2. From (60), for all $b>0$, a lower bound for r is given by r_0. For all $\eta\in(0,1)$ and all $r>0$, the solution to (59) is such that $b>0$. Hence, for all $\eta\in(0,1)$, the solution to (59)-(60) is such that $b>0$ and $r>r_0$. From (59), for all $r>0$, as η goes to 0, b goes to 0 and hence, from (60), $r\to r_0$. In the other polar case, $\eta\to 1$, from (60), $r\to r_0$. ■

APPENDIX B.
ADDITIONAL MATERIAL ON: “FRICTIONAL CAPITAL MARKET WITH KNOWLEDGE SPILLOVER”

B1. Tobin’s marginal q

I derive the expression for the marginal Tobin’s q in closed form.

B2. Value of intermediaries

I compute the market value of brokers-dealers.

B3. Market clearing

I describe an alternative approach to market clearing based on stocks rather than flows.

B4. Calibration strategy

I provide additional details on the calibration strategy

B5. Welfare

I derive the closed-form expression for the welfare gain from removing market frictions.

B1. Tobin's marginal q

One can use Lemma 2 to obtain the value of a marginal unit of capital at a firm, the Tobin's marginal q defined as $\upsilon'_{j,t}(k)$, along the equilibrium path.[28]

Lemma 6 ***(Tobin's marginal*** q***)*** *The Tobin's marginal* q *of a firm at time* t *with current productivity* A_j*,* $j \in \{1, ..., J\}$*, who accessed the market for the last time at* $t - \tau$ *with productivity* A_i*,* $i \in \{1, ..., J\}$*, is equal to:*

$$q_{ij}(\tau) = 1 + \frac{r}{r + \alpha(1-\eta)} \left[e^{(1-\beta)g\tau} \frac{\hat{A}_j}{\hat{A}_i} - 1 \right], \ \forall (i,j) \in \{1, ..., J\}^2, \forall \tau \in \mathbb{R}^+. \qquad (79)$$

Proof. The Tobin's marginal q of a firm with productivity type j and capital k is $\upsilon'_{j,t}(k)$. From (34), it is equal to

$$\upsilon'_{j,t}(k) = \frac{\beta \hat{A}_j}{r - (1-\beta)g + \alpha(1-\eta)} \left(\frac{K_t}{k} \right)^{1-\beta} + \frac{\alpha(1-\eta)}{\alpha(1-\eta) + r}.$$

Along the equilibrium path, the capital stock of a firm, k, is determined as follows. If the firm accessed the market at $t - \tau$ with productivity type i, then its capital is $k_{i,t-\tau}$ where $k_{i,t}/K_t$ is given by (37). Using that K_t grows at rate g,

$$\begin{aligned} \left(\frac{K_t}{k_{i,t-\tau}} \right)^{1-\beta} &= \left(\frac{e^{g\tau} K_{t-\tau}}{k_{i,t-\tau}} \right)^{1-\beta} \\ &= e^{(1-\beta)g\tau} \left(\frac{K_{t-\tau}}{k_{i,t-\tau}} \right)^{1-\beta} \\ &= \frac{e^{(1-\beta)g\tau} r \left[r + \alpha(1-\eta) - (1-\beta)g \right]}{\beta \left[r + \alpha(1-\eta) \right] \hat{A}_i}, \end{aligned}$$

where I used (37) to obtain the last equality. Substitute this expression for $(K_t/k_{i,t-\tau})^{1-\beta}$ into $\upsilon'_{j,t}(k)$ above to obtain:

$$q_{ij}(\tau) \equiv \upsilon'_{j,t}(k_{i,t-\tau}) = \frac{e^{(1-\beta)g\tau} r \hat{A}_j}{\left[r + \alpha(1-\eta) \right] \hat{A}_i} + \frac{\alpha(1-\eta)}{\alpha(1-\eta) + r}.$$

It can be rearranged to obtain (79). ■

The Tobin's q is exactly equal to one when $\tau = 0$ and $A_j = A_i$, which corresponds to the time at which a firm exits the capital market. Once a firm has readjusted its capital,

[28] For a recent study of the measurement of Tobin's marginal q, see Gala, Gomes, and Liu (2024). See their Table 4 for the distribution of the marginal q in the data.

its Tobin's q, $q_{jj}(\tau)$, increases over time, due to aggregate productivity rising, until the next idiosyncratic shock hits. At that point, the Tobin's q can increase or decrease. From (79),

$$q_{ij}(\tau) < 1 \Leftrightarrow \tau < \frac{\ln(\hat{A}_i) - \ln(\hat{A}_j)}{(1-\beta)g}.$$

The Tobin's q is less than one if the firm's current idiosyncratic productivity is lower than its value at the time the firm accessed the market, $A_j < A_i$, and that time is not too far away. If $A_j > A_i$, the Tobin's q is greater than one irrespective of τ.

References

[1] Gala, Vito, João Gomes, and Tong Liu (2024). "Marginal q". Working Paper.

B2. Value of intermediaries

The market value of a broker, w_t, solves

$$rw_t = \alpha\Phi_t + \dot{w}_t \quad \text{where } \Phi_t \equiv \sum_{(i,j)\in\{1,\ldots,J\}^2} \int_{\mathbb{R}^+} \phi_t(i,j,\tau)\gamma(i,j,\tau)d\tau \tag{80}$$

is the broker's expected revenue from a random meeting with a firm and $\phi_t(i,j,\tau)$ denotes the broker's revenue in a meeting with a firm of type (i,j,τ). From the bargaining solution, (8),

$$\phi_t(i,j,\tau) = \eta\left[\upsilon_{j,t}(k_{j,t}) - (k_{j,t} - k_{i,t-\tau}) - \upsilon_{j,t}(k_{i,t-\tau})\right]. \tag{81}$$

The broker appropriates a fraction, η, of the increase in the firm's financial value from accessing the capital market. Note that there are two reasons for the firm to readjust its capital from $k_{i,t-\tau}$ to $k_{j,t}$. First, its idiosyncratic productivity might have changed since its last access to the market, $i \neq j$. Second, the steady increase in aggregate productivity raises firms' demand for capital over time.

Lemma 7 *The value of a broker is*

$$w_t = \frac{\alpha\Phi_t}{r-g}, \tag{82}$$

where

$$\Phi_t = \eta\left\{\frac{\sum_j\left[(\lambda+\beta g)\hat{A}_j - \frac{\alpha\lambda}{\alpha+\beta g}\bar{A}\right]\pi_j\tilde{k}_j^\beta}{\left[r-(1-\beta)g+\alpha(1-\eta)\right](\alpha+\lambda+\beta g)} - \frac{rg}{\alpha\left[\alpha(1-\eta)+r\right]}\right\}K_t. \tag{83}$$

In order to get some intuition for the expression for Φ_t in (83), suppose there is no productivity growth, $g=0$, and $J=2$. The average profits of the broker reduce to

$$\Phi_t = \eta\left(\frac{\lambda\pi_1\pi_2}{\alpha+\lambda}\right)\frac{\left(\hat{A}_2-\hat{A}_1\right)\left(\tilde{k}_2^\beta-\tilde{k}_1^\beta\right)}{r+\alpha(1-\eta)}K_t.$$

In the capital market, brokers reallocate $\tilde{k}_2-\tilde{k}_1$ units of capital from the type-1 firms who hold $\tilde{k}_2$ to the type-2 firms who hold $\tilde{k}_1$. The steady-state measure of firms with idiosyncratic productivity A_1 holding k_2 units of capital is equal to $\lambda\pi_1\pi_2/(\alpha+\lambda)$, which is also the measure of A_2-firms holding k_1. Thus, $\lambda\pi_1\pi_2/(\alpha+\lambda)$ can be interpreted as the flow of low- and high-productivity firms that are matched to reallocate their capital. The instantaneous output gain from this reallocation, normalized by the aggregate capital stock, is

$$\hat{A}_2\tilde{k}_2^\beta + \hat{A}_1\tilde{k}_1^\beta - \left(\hat{A}_2\tilde{k}_1^\beta + \hat{A}_1\tilde{k}_2^\beta\right) = \left(\hat{A}_2-\hat{A}_1\right)\left(\tilde{k}_2^\beta-\tilde{k}_1^\beta\right).$$

Finally, the effective discount rate is $r+\alpha(1-\eta)$.

Proof of Lemma 7. Using the closed-form solution for $v_{j,t}(k_{j,t})$ given by (34), $\phi_t(i,j,\tau)$ given by (81) can be reexpressed as

$$\phi_t(i,j,\tau)=\eta\left[\frac{\hat{A}_j K_t^{1-\beta}\left(k_{j,t}^{\beta}-k_{i,t-\tau}^{\beta}\right)}{r-(1-\beta)g+\alpha(1-\eta)}-\frac{r(k_{j,t}-k_{i,t-\tau})}{\alpha(1-\eta)+r}\right]. \tag{84}$$

Using that $k_{i,t-\tau}=e^{-g\tau}k_{i,t}$ and the notations $\tilde{k}_j\equiv k_{j,t}/K_t$ and $\tilde{\phi}(i,j,\tau)\equiv\phi_t(i,j,\tau)/K_t$,

$$\tilde{\phi}(i,j,\tau)=\eta\left[\frac{\hat{A}_j\left(\tilde{k}_j^{\beta}-e^{-\beta g\tau}\tilde{k}_i^{\beta}\right)}{r-(1-\beta)g+\alpha(1-\eta)}-\frac{r(\tilde{k}_j-e^{-g\tau}\tilde{k}_i)}{\alpha(1-\eta)+r}\right], \tag{85}$$

and $\tilde{k}_j$ is given by (37). Using that $\dot{w}=gw$, w_t given by (80) solves

$$w_t=\frac{\alpha\Phi_t}{r-g}, \tag{86}$$

where a broker's expected profits from a meeting with a firm picked at random in the population of firms are

$$\Phi_t\equiv\int_0^{+\infty}\sum_{(i,j)\in\{1,\dots,J\}^2}\phi_t(i,j,\tau)\gamma(i,j,\tau)d\tau,$$

where $(i,j,\tau)\in\{1,\dots,J\}^2\times\mathbb{R}^+$ is a triplet composed of the current productivity of the firm, j, the time that has elapsed since its last access to the market, τ, and its productivity at that time, i, and $\gamma(i,j,\tau)$ is the density measure of firms,

$$\gamma(i,j,\tau)=\alpha e^{-\alpha\tau}\pi_i\left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}}+(1-e^{-\lambda\tau})\pi_j\right].$$

In order to compute Φ_t, we decompose it as follows:

$$\Phi_t=\eta\left[\frac{Q_1-Q_2}{r-(1-\beta)g+\alpha(1-\eta)}-\frac{r(Q_3-Q_4)}{\alpha(1-\eta)+r}\right]K_t, \tag{87}$$

where:

1. Q_1 is given by:

$$\begin{aligned}
Q_1 &\equiv \int_0^{+\infty}\sum_{(i,j)\in\{1,\dots,J\}^2}\hat{A}_j\tilde{k}_j^{\beta}\gamma(i,j,\tau)d\tau\\
&= \int_0^{+\infty}\sum_{(i,j)}\hat{A}_j\tilde{k}_j^{\beta}\alpha e^{-\alpha\tau}\pi_i\left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}}+(1-e^{-\lambda\tau})\pi_j\right]d\tau\\
&= \int_0^{+\infty}\sum_j\pi_j\hat{A}_j\tilde{k}_j^{\beta}\alpha e^{-\alpha\tau}d\tau\\
&= \sum_j\pi_j\hat{A}_j\tilde{k}_j^{\beta}.
\end{aligned}$$

We move from the second line to the third line by taking the sum over $i \in \{1, ..., J\}$ and use the fact that

$$\sum_i \pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] = \pi_j e^{-\lambda\tau} + (1-e^{-\lambda\tau})\pi_j = \pi_j.$$

We move from the third line to the fourth by changing the order of the sum and integral and noticing that $\int_0^{+\infty} \alpha e^{-\alpha\tau} d\tau = 1$.

2. Q_2 is given by:

$$\begin{aligned}
Q_2 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} e^{-\beta g\tau} \hat{A}_j \tilde{k}_i^\beta \gamma(i,j,\tau) d\tau \\
&= \int_0^{+\infty} \sum_{(i,j)} e^{-\beta g\tau} \hat{A}_j \tilde{k}_i^\beta \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] d\tau \\
&= \int_0^{+\infty} \sum_i \alpha e^{-(\alpha+\beta g)\tau} \left[e^{-\lambda\tau}\hat{A}_i + (1-e^{-\lambda\tau})\bar{A}\right] \pi_i \tilde{k}_i^\beta d\tau \\
&= \frac{\alpha}{\alpha+\lambda+\beta g} \sum_i \pi_i \left[\hat{A}_i + \frac{\lambda}{\alpha+\beta g}\bar{A}\right] \tilde{k}_i^\beta.
\end{aligned}$$

We move from the second line to the third line by taking the sum over $j \in \{1, ..., J\}$ and use the fact that:

$$\sum_j \hat{A}_j \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] = e^{-\lambda\tau}\hat{A}_i + (1-e^{-\lambda\tau}) \sum_j \hat{A}_j \pi_j.$$

Moreover, $\sum_j \hat{A}_j \pi_j = \bar{A}$. We move from the third line to the fourth line by using that

$$\begin{aligned}
\int_0^{+\infty} \alpha e^{-(\alpha+\beta g)\tau} e^{-\lambda\tau} \hat{A}_i d\tau &= \frac{\alpha}{\alpha+\lambda+\beta g}\hat{A}_i \\
\int_0^{+\infty} \alpha e^{-(\alpha+\beta g)\tau} (1-e^{-\lambda\tau}) \bar{A} d\tau &= \frac{\alpha\lambda}{(\alpha+\beta g)(\alpha+\lambda+\beta g)}\bar{A}.
\end{aligned}$$

3. Q_3 is given by:

$$\begin{aligned}
Q_3 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} \tilde{k}_j \gamma(i,j,\tau) d\tau \\
&= \int_0^{+\infty} \sum_{(i,j)} \tilde{k}_j \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] d\tau \\
&= \int_0^{+\infty} \sum_j \alpha e^{-\alpha\tau} \pi_j \tilde{k}_j d\tau \\
&= \sum_j \pi_j \tilde{k}_j
\end{aligned}$$

4. Q_4 is given by:

$$
\begin{aligned}
Q_4 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,\ldots,J\}^2} e^{-g\tau}\tilde{k}_i \gamma(i,j,\tau)d\tau \\
&= \int_0^{+\infty} \sum_{(i,j)} e^{-g\tau}\tilde{k}_i \alpha e^{-\alpha\tau}\pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] d\tau \\
&= \int_0^{+\infty} \sum_i e^{-g\tau}\alpha e^{-\alpha\tau}\pi_i \tilde{k}_i d\tau \\
&= \frac{\alpha}{\alpha+g}\sum_i \pi_i \tilde{k}_i
\end{aligned}
$$

From the expressions above,

$$
\begin{aligned}
Q_1 - Q_2 &= \frac{1}{\alpha+\lambda+\beta g}\sum_j \left[(\lambda+\beta g)\,\hat{A}_j - \frac{\alpha\lambda}{\alpha+\beta g}\bar{A}\right]\pi_j \tilde{k}_j^{\beta} \\
Q_3 - Q_4 &= \frac{g}{\alpha+g}\sum_j \pi_j \tilde{k}_j = \frac{g}{\alpha},
\end{aligned}
$$

where we used that, by market clearing, $\sum_j \pi_j \tilde{k}_j = 1 + g/\alpha$. We substitute $Q_1 - Q_2$ and $Q_3 - Q_4$ by their expressions into (87) to obtain (83).

■

B3. Market clearing

In the main text, the market-clearing condition is expressed in terms of flows. Alternatively, the market-clearing condition can be written in terms of stocks as:

$$\sum_{(i,j)\in\{1,\ldots,J\}^2} \int_{\mathbb{R}^+} e^{-g\tau}\tilde{k}_{i,t}\gamma(i,j,\tau)d\tau = 1, \tag{88}$$

where $e^{-g\tau}\tilde{k}_{i,t}$ is the capital held by a firm, normalized by K_t, who had access to the market at time $t-\tau$ and had productivity type i at that time. The right side corresponds to the aggregate capital held by all firms normalized by K_t, which is equal to one. Substituting $\gamma(i,j,\tau)$ by its expression given by (38), it can be shown after some calculation that (88) reduces to (41).

B4. Calibration strategy

The unit of time is a year. The rate of time preference is set to $\rho = 0.01$ and the intertemporal elasticity of substitution to $\theta^{-1} = 0.5$. I use the average productivity, $\bar{A}$, to target a growth rate of consumption of $g = 2\%$. As a result, the real interest rate is $r = \theta g + \rho = 5\%$. The technology is $A_j K^{1-\beta} k^{\beta}$ where β is chosen to be approximately equal to the capital share, i.e., $\beta = 0.4$.[29]

The process for the idiosyncratic productivity shocks is analogous to the one in Cui (2022). There are two levels of productivity, $A_L < A_H$, such that $A_H/A_L = 1.64$, drawn with equal probabilities, $\pi_L = \pi_H = 0.5$. I set $\lambda = 0.5$. Hence, on average, a firm changes productivity every $(\lambda\pi_j)^{-1} = 4$ years. The probability that a firm does not draw a new productivity value within a year is $e^{-\lambda\pi_H} = e^{-\lambda\pi_L} = 0.78$, which is in the same ballpark as the measure of yearly persistence of plant-level productivity shocks in the US reported by Cui (2022) and references therein.

Capital reallocation is defined as the sales of existing capital across all firms. In the model, it is ϱK where

$$\varrho \equiv \alpha \sum_{(i,j)\in\{L,H\}^2} \int_0^{+\infty} \gamma(i,j,\tau) \max\{e^{-g\tau}\tilde{k}_i - \tilde{k}_j, 0\} d\tau.$$

The flow of firms who access the market per unit of time is α. The sale of capital by a firm of type j who last accessed the market at time $t-\tau$ with productivity i is equal to $e^{-g\tau}\tilde{k}_i - \tilde{k}_j$ if this quantity is positive and zero otherwise. The term, $e^{-g\tau}\tilde{k}_i$, corresponds to the capital held by the firm (normalized by the aggregate capital stock, K_t) while $\tilde{k}_j$ is its desired capital stock. Only type-L firms who were previously type-H when they accessed the market at time $\tau < \hat{\tau} \equiv \ln(\tilde{k}_H/\tilde{k}_L)/g$ sell some capital. Hence,

$$\varrho = \alpha \int_0^{\hat{\tau}} \gamma(H,L,\tau) \left(e^{-g\tau}\tilde{k}_H - \tilde{k}_L\right) d\tau.$$

Substitute $\gamma(H,L,\tau)$ by its expression given by (38), i.e., $\alpha e^{-\alpha\tau}\pi_H\pi_L(1-e^{-\lambda\tau})$, and use that $k_H = e^{\hat{\tau}g}k_L$ to obtain

$$\varrho = \alpha\pi_H\pi_L\tilde{k}_L \int_0^{\hat{\tau}} \alpha e^{-\alpha\tau}(1-e^{-\lambda\tau})\left[e^{g(\hat{\tau}-\tau)} - 1\right] d\tau.$$

According to Eistfeldt and Shi (2018), capital reallocation represents about 2% of firms total assets annually. I interpret total assets as K_t and I target $\varrho = 0.02$. The parameter to

[29] In the model, there is no labor so that all the income is capital income. However, the production function could be generalized so as to incorporate a labor input, ℓ, supplied inelastically by households.

achieve this target is broker's bargaining power. Indeed, η affects the dispersion of asset demands, $\tilde{k}_H/\tilde{k}_L$, and the measure of reallocation, ϱ.

In order to calibrate α, I resort to different sources. In the model, α is the rate at which a firm can sell its excess capital. Exapro, an online marketplace for used industrial equipment, reports that "on average machines were sold on Exapro after 8 months since being published in the catalog."[30] Southern Fabricating Machinery Sales, a machinery broker, reports that the time to sell varies from 0-30 days for newer machinery and 60-90 days for older machinery.[31] According to the owner of Surplus Record, a leading business directory of used machine tools, machinery, and industrial equipment, the turnover (defined as the number of removed listings in a given period of time relative to the total number of listings) of lathe machines is about 9% per month.[32] One can interpret this turnover rate as α, the rate at which firms can sell their capital. Across machines that are traded frequently, the turnover ranges from 5% to 10% each month. In order to obtain search frictions in the ballpark of these numbers, I set $\alpha = 2$. On average, it takes 6 months for a firm to sell the capital it does not need.[33]

[30] https://hub.exapro.com/how-long-does-it-take-to-sell-used-machinery

[31] https://www.southernfabsales.com/blog/the-art-of-valuing-used-machinery

[32] According to Wikipedia, "a lathe is a machine tool that rotates a workpiece about an axis of rotation to perform various operations such as cutting, sanding, knurling, drilling, deformation, facing, threading and turning, with tools that are applied to the workpiece to create an object with symmetry about that axis." I thank Tom Scanlan for providing me with the turnover data and explanations.

[33] Gavazza (2016) in his study of the market for business aircrafts estimates that an aircraft stays on the market approximately eight months before a low-valuation owner is able to sell it and that, on average, it takes slightly more than four months for a dealer to find a buyer.

B5. Welfare

Along a balanced growth path, consumption is $C_t = Y_t - \dot{K}_t = (TFP - g)\, K_t$, and the household lifetime utility is

$$U(C_t) \equiv \int_0^{+\infty} e^{-\rho t} \frac{(C_t)^{1-\theta}}{1-\theta} dt = \frac{K_0^{1-\theta}}{1-\theta} \frac{(TFP - g)^{1-\theta}}{[\rho - (1-\theta) g]}, \tag{89}$$

where I used that $C_t = C_0 e^{gt}$ with $C_0 = (TFP - g)\, K_0$. Market frictions affect both the level of the aggregate output, TFP, and its growth rate, g, possibly in opposite directions.

The welfare gain from removing market frictions, Δ, solves $U(C_t) = U\left[C_t^{ce}(1-\Delta)\right]$, where C_t^{ce} is the time-path for consumption at the frictionless competitive equilibrium. Following the same reasoning as in (89),

$$U\left[C_t^{ce}(1-\Delta)\right] = (1-\Delta)^{1-\theta} \frac{K_0^{1-\theta}}{1-\theta} \frac{(TFP^* - g^{ce})^{1-\theta}}{[\rho - (1-\theta) g^{ce}]},$$

where I used that the aggregate output per unit of capital in the competitive economy is TFP^*, given by (55), and the growth rate is $g^{ce} = (r^{ce} - \rho)/\theta$, where $r^{ce} = \beta r^*$ by (47). Hence,

$$1 - \Delta = \left[\frac{\rho - (1-\theta) g^{ce}}{\rho - (1-\theta) g} \right]^{\frac{1}{1-\theta}} \frac{TFP - g}{TFP^* - g^{ce}}.$$

APPENDIX C

C1. Distribution of capital across firms

I characterize the distribution of capital across firms in the model with linear technologies of Section 3.

C2. Generalized idiosyncratic productivity shocks

I consider a general continuous-time Markov chain for the process of productivity shocks and I show that capital might not always reallocated toward firms with the highest instantaneous productivity.

C3 Transitional dynamics

I characterize transitional dynamics in the model with linear technologies of Section 3.

C4. Frictions in the primary capital market

In the benchmark model, new capital is sold instantly at the competitive inter-broker price. In this Appendix, I make the primary market for capital goods frictional: there are search and bargaining frictions to sell the new capital.

C5. Endogenous bargaining power of brokers

I endogenize the market structure by assuming that brokers can choose their bargaining power at some cost.

C6. When the knowledge externality does not sustain growth

I consider a version of the model of Section 4 where the learning-by-investing externality is not strong enough to sustain long-term growth but I show that the model can be useful to explain incomes differences.

C7. Externalities and growth

I propose an alternative description of the externality that generates long-term growth. In this version, firms choose an input that is a complement to capital in production. The externality operates through that input.

C8. Embodied technological progress

I consider a version of the model where learning-by-doing lowers the cost of production of capital producers.

C9. Alternative calibration

I describe an alternative calibration where I reduce the strength of the learning-by-investing externality and I show that the main insights are robust.

C10. Growth, money, and capital reallocation

I extend the model of Section 3 to introduce liquidity constraints. Firms need to hold costly liquidity in order to purchase capital.

C1. Distribution of capital across firms

We characterize how capital is distributed across firms. We start with firms with the highest productivity level, H. There is a measure π_H of such firms. They hold a positive capital stock if their productivity type was also at the highest at the time of their last participation in the capital market. Suppose they participated in the market at time $t-\tau$ with $\tau > 0$. How much capital did they leave the market with? The quantity of capital in the market available to be allocated to firms over a time interval of length dt is

$$\text{capital supply at time } t-\tau = \overbrace{\alpha dt K_{t-\tau}}^{\text{second-hand capital}} + \overbrace{\dot{K}_{t-\tau} dt}^{\text{new capital}}.$$

There is a measure αdt of firms who participated the capital market at time $t-\tau$. By the law of large number, their average capital stock is equal to the aggregate capital stock, $K_{t-\tau}$. In addition, there is a production of new capital, $\dot{K}_{t-\tau}dt$, that is instantly traded in the capital market. The measure of investors of the highest productivity type is $\alpha dt \pi_H$. Assuming a symmetric equilibrium where firms of type H leave the market with the same capital, their capital holding is

$$k_{H,t-\tau} = \frac{\alpha K_{t-\tau} + \dot{K}_{t-\tau}}{\alpha \pi_H}. \tag{90}$$

We focus on balanced-growth paths where aggregate variables grow at the same rate g. Hence, $K_{t-\tau} = e^{-g\tau}K_t$ and $\dot{K}_{t-\tau} = gK_{t-\tau}$, which gives

$$k_{H,t-\tau} = \frac{(\alpha+g)e^{-g\tau}K_t}{\alpha \pi_H}. \tag{91}$$

The density measure of firms with productivity type H and capital $k_{H,t-\tau}$ is

$$\gamma_H(\tau) = \alpha e^{-\alpha\tau}\pi_H \left[e^{-\lambda\tau} + (1-e^{-\lambda\tau})\pi_H\right] \quad \text{for all } \tau > 0. \tag{92}$$

This expression is explained as follows. The density measure of the firms who contacted the market at time $t-\tau$ is $\alpha e^{-\alpha\tau}$. Firms that own $k_{H,t-\tau} > 0$ units of capital must have been of type H the last time they participated in the market since otherwise they would have liquidated their capital. There is a fraction π_H of such firms. These firms are still of type H at time t if: either they did not receive a productivity shock during the interval of length τ, with probability, $e^{-\lambda\tau}$; of they received at least one productivity shock, with probability $1-e^{-\lambda\tau}$, and the realization of the last one was H with probability π_H.

Consider next firms with productivity L. Those firms have a positive capital stock equal to $k_{h,t-\tau}$ if they contacted the market at time $t-\tau$ and had a productivity type equal to H. By a similar reasoning as above, the density measure of such firms is

$$\gamma_L(\tau) = \alpha e^{-\alpha\tau}\pi_H(1-e^{-\lambda\tau})\pi_L. \tag{93}$$

The difference relative to (92) is that a firm who is of type L and owns a positive capital stock at time t must have received at least one productivity shock between $t-\tau$ and t, with probability $1-e^{-\lambda\tau}$, and the realization of the last shock is L. The next Lemma makes use of the density functions, (92) and (93), to compute the cumulative distribution, Υ.

Lemma 8 ***(Distribution of capital across firms.)*** *The measure of high-productivity firms who owns no more than k units of capital at time t is*

$$\Upsilon_{H,t}(k)=\pi_H\left\{1-\left(\frac{\alpha+\lambda\pi_H}{\alpha+\lambda}\right)+\frac{\pi_L\alpha}{\alpha+\lambda}\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\alpha+\lambda}{g}}+\pi_H\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\alpha}{g}}\right\}, \quad (94)$$

for all $k\leq(\alpha+g)K_t/\alpha\pi_H$.

The measure of low-productivity firms who owns no more than k units of capital at time t is:

$$\Upsilon_{L,t}(k)=\pi_L\left\{1-\pi_H\left\{\frac{\alpha}{\alpha+\lambda}\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\lambda+\alpha}{g}}-\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\alpha}{g}}+\frac{\lambda}{\alpha+\lambda}\right\}\right\}, \quad (95)$$

for all $k\leq(\alpha+g)K_t/\alpha\pi_H$.

Proof. The cumulative measure, $\Upsilon_{H,t}(k)$, is given by:

$$\Upsilon_{H,t}(k)=\pi_H-\int_0^{\ln\left[\frac{(\alpha+g)K_t}{\alpha\pi_H k}\right]^{\frac{1}{g}}}\gamma_H(\tau)d\tau \quad \forall k\in(0,k_{H,t}]\,. \quad (96)$$

The measure of all firms of type H is π_H. Assuming the growth rate is positive, $g>0$, the maximum capital held by a firm of type H at time t is $k_{H,t}=(\alpha+g)K_t/(\alpha\pi_H)$. So, for all $k\geq k_{H,t}$, $\Upsilon_{H,t}(k)=\pi_H$. If $k<k_{H,t}$, we substract the measure of measure who owns a capital stock between k and $k_{H,t}$. These firms last accessed the market after time $t-\tau$ where τ is the solution to $k_{H,t-\tau}=k$, i.e., $\tau=\ln\left[(\alpha+g)K_t/\alpha\pi_H k\right]^{1/g}$. Substitute $\gamma_H(\tau)$ from its expression given by (92) into (96) and rearrange to obtain (94).

Similarly, in order to compute $\Upsilon_{L,t}(k)$, we integrate $\gamma_L(\tau)$ as follows:

$$\Upsilon_{L,t}(k)=\pi_L-\int_0^{\ln\left[\frac{(\alpha+g)K_t}{\alpha\pi_H k}\right]^{\frac{1}{g}}}\gamma_L(\tau)d\tau \quad \forall k\in(0,k_{H,t}]\,.$$

This gives (95). ■

From (94) there is a mass of firms of type H with $k=0$ which is equal to

$$\Upsilon_{H,t}(0^+)=\pi_H-\pi_H\left(\frac{\alpha+\lambda\pi_H}{\alpha+\lambda}\right)=\pi_L\pi_H\left(\frac{\lambda}{\alpha+\lambda}\right).$$

It is explained as follows. A measure π_L of firms had a low productivity at their last contact with the market. Among those firms, a fraction $\lambda/(\alpha+\lambda)$ experienced at least one productivity shock between $t-\tau$ and t and the realization of the last shock was H with probability π_H. Similarly, the measure of type-L firms with zero capital is

$$\Upsilon_{L,t}(0^+) = \pi_L \left(\frac{\alpha + \lambda \pi_L}{\alpha + \lambda} \right).$$

C2. Generalized idiosyncratic productivity shocks

In this Appendix, I generalize the stochastic process driving firms' idiosyncratic productivity. I assume that there are $J \geq 2$ distinct realizations, indexed by $j \in \{1, ..., J\}$, for a firm's idiosyncratic productivity. Productivities are ranked as follows: $A_1 < A_2 < ... < A_J$. Suppose that A_j follows a continuous-time Markov chain with generator matrix $\mathbf{\Lambda} = [\lambda_{ji}]$ where λ_{ji} is the rate at which productivity transitions from j to $i \neq j$ and $\lambda_{jj} = -\sum_{i \neq j} \lambda_{ji}$.

In the main text, I specialized the analysis to $J = 2$. If shocks are i.i.d. so that $\lambda_{ji} = \lambda \pi_i$, with $\lambda > 0$ and $\pi_i \in [0, 1]$, for all i and j, then the restriction to $J = 2$ is with no loss in generality. Otherwise, if shocks are not i.i.d., some insights from $J = 2$ do not carry over to the case where $J \geq 3$. In particular, it is not necessarily the case that capital is reallocated toward the type-J firms with the highest instantaneous productivity.

The Tobin's q of a firm with idiosyncratic productivity j solves

$$rq_j = A_j + \sum_{i \in \{1,...,J\}} \lambda_{ji} q_i + \alpha(1 - \eta)(1 - q_j). \tag{97}$$

The first term on the right side is the flow output generated by one unit of capital. The second term represents the capital gains or losses as productivity changes. In order to see this, note that

$$\begin{aligned} \sum_{i \in \{1,...,J\}} \lambda_{ji} q_i &= \sum_{i \in \{1,...,J\} \backslash \{j\}} \lambda_{ji} q_i + \lambda_{jj} q_j \\ &= \sum_{i \in \{1,...,J\} \backslash \{j\}} \lambda_{ji} q_i - \sum_{i \in \{1,...,J\} \backslash \{j\}} \lambda_{ji} q_j \\ &= \sum_{i \in \{1,...,J\} \backslash \{j\}} \lambda_{ji} (q_i - q_j), \end{aligned}$$

where, in order to obtain the second equality, I used the definition of λ_{jj}. The last term on the right side of (97) corresponds to the gain from liquidating the capital when the firm has access to the market. Here, I used the observation that $q_j \leq 1$ for all j so that the capital market can clear. The equations (97) can be rewritten in a matrix form as

$$[r + \alpha(1 - \eta)] \mathbf{q} = \mathbf{A} + \mathbf{\Lambda q} + \alpha(1 - \eta)\mathbf{1}. \tag{98}$$

Solving for $\mathbf{q}$,

$$\mathbf{q} = \{[r + \alpha(1 - \eta)] \mathbf{I} - \mathbf{\Lambda}\}^{-1} [\mathbf{A} + \alpha(1 - \eta)\mathbf{1}]. \tag{99}$$

For market clearing, the firms with the highest q_j must be indifferent between acquiring additional capital or not. Hence,

$$\max_{j \in \{1,...,J\}} \{q_j\} = 1. \tag{100}$$

An equilibrium is a $r > \rho$ and $\mathbf{q} \in \mathbb{R}^J$ solution to (99) and (100).

As an example, suppose $J = 3$, $\lambda_{31} > 0$, $\lambda_{12} > 0$, $\lambda_{23} > 0$, and all the other $\lambda_{ij} = 0$. Thus, type-1 firms become type-2 firms who become type-3 firms who then return to type 1. I construct an equilibrium where $\max\{q_j\} = q_2 = 1$.

Proposition 11 *Suppose $A_1 > \rho$ and $\theta > 1$. For any $\alpha < +\infty$, if η is sufficiently close to 1 and λ_{31} is sufficiently large, then there exist an equilibrium with $q_1 < 1$, $q_3 < 1$, and $q_2 = 1$.*

Proof. Using that $q_2 = 1$, the HJB equations can be rewritten as:

$$\begin{aligned} rq_1 &= A_1 + \lambda_{12}(1 - q_1) + \alpha(1-\eta)(1-q_1) \\ r &= A_2 + \lambda_{23}(q_3 - 1) \\ rq_3 &= A_3 + \lambda_{31}(q_1 - q_3) + \alpha(1-\eta)(1-q_3) \end{aligned}$$

It can be solved for q_1 and q_3 to obtain:

$$\begin{aligned} q_1 &= 1 - \frac{r - A_1}{\lambda_{12} + \alpha(1-\eta) + r} \\ q_3 &= [r + \alpha(1-\eta)]^{-1} \left\{ A_3 + \lambda_{31} \frac{(A_1 - A_3)[\lambda_{12} + \alpha(1-\eta) + r] + \lambda_{12}(r - A_1)}{[r + \lambda_{31} + \alpha(1-\eta)][\lambda_{12} + \alpha(1-\eta) + r]} + \alpha(1-\eta) \right\} \end{aligned}$$

Suppose $\eta \approx 1$ and $\lambda_{31} \to +\infty$. Then,

$$\begin{aligned} q_1 &= q_3 = \frac{\lambda_{12} + A_1}{\lambda_{12} + r} \\ r &= A_2 + \lambda_{23} \frac{A_1 - r}{\lambda_{12} + r} \end{aligned}$$

It can be checked that $r \in (A_1, A_2)$ and hence $q_1 = q_3 < 1$. Since $r > A_1$, $g = (r - \rho)/\theta > 0$. Moreover, since $\theta > 1$, $r - g > 0$. ■

If brokers' bargaining power is close to 1 and the transition from type 3 to type 1 is fast, then capital operated by type-2 firms has the highest market value. Moreover, if α is large, most of the capital will be held by type-2 firms even though both the first-best and the constrained-efficient allocations prescribe that capital should be held by type-3 firms. So, brokers' bargaining power can affect directly both the static misallocation of capital and the dynamic misallocation.

C3. Transitional dynamics.

I now describe transitional dynamics from arbitrary initial conditions, $(K_{L,0}, K_{H,0})$. First, the equations defining the Tobin's q, (11), do not depend on the composition of the capital. Hence, q_j and r (which is determined from $q_H = 1$) jump instantly to their new balanced-growth path values at time $t = 0$. The growth rate of consumption, $g = (r - \rho)/\theta$, is also constant on the transitional path.

Consider next the time-paths of the capital stocks. I denote $k_t \equiv K_t/(e^{gt}K_0)$ and $k_{L,t} \equiv K_{L,t}/\left(e^{gt}K_0\right)$. The detrended capital stocks, k_t and $k_{L,t}$, evolve according to:

$$\dot{k}_t = (A_H - g)k_t - (A_H - A_L)k_{L,t} - c_0 \tag{101}$$

$$\dot{k}_{L,t} = \lambda\pi_L k_t - (\alpha + g + \lambda)\, k_{L,t}, \tag{102}$$

where $c_0 \equiv C_0/K_0$. The initial conditions are $k_0 = 1$ and $k_{L,0} = K_{L,0}/K_0$. Equation (101) is obtained from $\dot{K}_t = Y_t - C_t$, where $Y_t = A_H K_{H,t} + A_L K_{L,t}$, while (102) is obtained from an ODE analogous to (16), $\dot{K}_{L,t} = \lambda\pi_L K_{H,t} - (\lambda\pi_H + \alpha)K_{L,t}$.

Lemma 9 *Suppose the conditions in Proposition 2 for the existence of a balanced-growth path equilibrium hold. The stationary solution to (101)-(102) is a saddle point.*

Proof. The Jacobian matrix associated with (101)-(102) is

$$M_J \equiv \begin{pmatrix} A_H - g & -(A_H - A_L) \\ \lambda\pi_L & -(\alpha + g + \lambda) \end{pmatrix}.$$

The determinant is

$$\det M_J = -\left(A_H - g\right)(\alpha + g + \lambda) + \lambda\pi_L\left(A_H - A_L\right).$$

From (**??**), in any balanced-growth-path equilibrium,

$$\frac{C_0}{K_0} = A_H - \left(A_H - A_L\right)\hat{k}_L - g$$
$$= A_H - g - \left(A_H - A_L\right)\frac{\lambda\pi_L}{\lambda + \alpha + g} > 0,$$

where the second equality is obtained by substituting $\hat{k}_L$ by its expression given by (17). It follows that $\det M_J < 0$. Hence, the solution to $\dot{k}_t = \dot{k}_{L,t} = 0$ is a saddle point. ■

From (102), the k_L-isocline given by $\dot{k}_{L,t} = 0$ is such that

$$\frac{k_{L,t}}{k_t} = \frac{\lambda\pi_L}{\alpha + g + \lambda} = \hat{k}_L.$$

From (101), the k-isocline given by $\dot{k}_t = 0$ is such that

$$k_{L,t} = \frac{(A_H - g)k_t - c_0}{(A_H - A_L)}.$$

So, both isoclines have a positive slope but, provided a balanced-growth path exists, the k-isocline is steeper. The phase diagram is represented in Figure 11. The endogenous variable, c_0, is determined so that the initial condition, $(1, k_{L,0})$, is located on the linear saddle path. In Figure 11, the initial condition for the composition of the capital stock is such that $k_{L,0} > \hat{k}_L$. Along the transition, K_t and $K_{L,t}$ approach their balanced growth path asymptotically and the ratio, $K_{L,t}/K_t$, decreases and tends to $\hat{k}_L$.

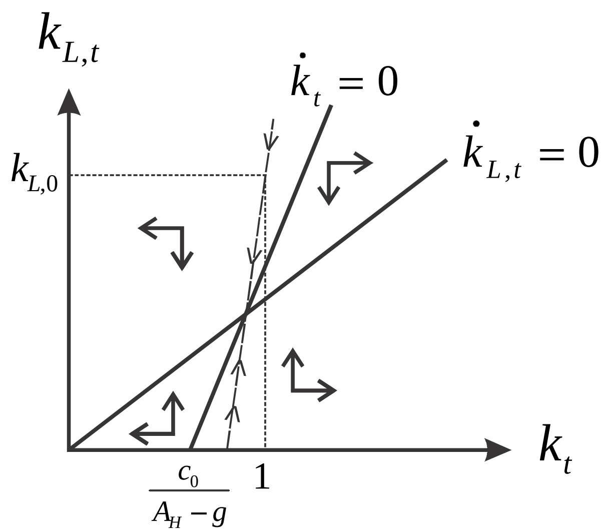

Figure 11: Transition dynamics

C4. Frictions in the primary capital market

In the main text, I have assumed that the newly produced capital was sold to brokers instantly at the competitive price, $p_K = 1$. In reality, even though capital producers have "*embraced a lean manufacturing model focused on matching production to demand*" (Darmouni and Sutherland, 2024), there are delays for new equipment to reach firms.[34] Therefore, I now relax the assumption that new capital is sold instantly and I make the capital market a two-sided brokered market as shown in Figure 12. There is a unit measure of *primary brokers* who purchase capital goods from capital producers. The meeting rate between capital producers and primary brokers is α_0. The bargaining power of primary brokers is $\eta_0 \in [0, 1]$. In the previous section, $\eta_0 = 0$ and $\alpha_0 = +\infty$. The secondary capital market is as described in the previous section. Firms meet *secondary brokers*, who have bargaining power η, at rate α.

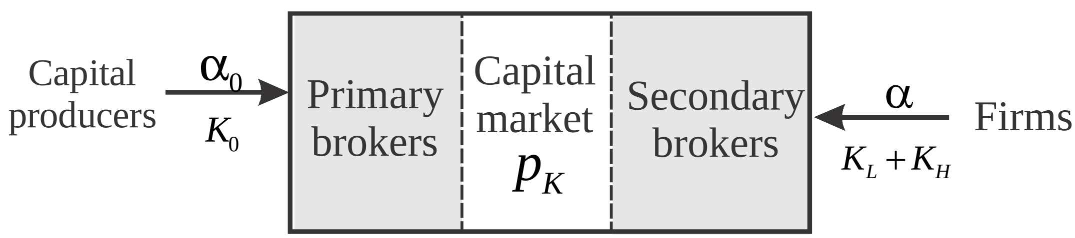

Figure 12: Capital market structure

The value of idle and employed capital

In order to determine the value of idle capital, I conjecture that the expected discounted profits of a capital producer with $k \geq 0$ units of idle capital can be expressed as $q_0 k$ where $q_0 \in \mathbb{R}^+$ is independent of k. Consider first the determination of the terms of trade in a meeting between a capital producer with k units of capital and a primary broker. The outcome of the negotiation is a pair, $(k^s, \phi) \in [0, k] \times \mathbb{R}_+$, that specifies the quantity sold, $k^s \leq k$, and the payment to the intermediary. The surplus of the capital producer is

$$q_0(k - k^s) + p_K k^s - \phi - q_0 k = (p_K - q_0)k^s - \phi,$$

where p_K is the price at which brokers trade capital and q_0 is the value to the capital producer of one unit of idle capital. The first term on the left side is the value of $k - k^s$ units of unsold

[34] Darmouni and Sutherland (2024) use the COVID episode to illustrate how supply chain disruptions can make new capital hard to find.

capital. The following two terms, $p_K k^s - \phi$, represent the revenue from the sale to the broker. The last term is the disagreement point if no trade takes place, in which case the value of the capital producer is $q_0 k$. The surplus of the broker is ϕ. As long as $p_K \geq q_0$, $k^s = k$ is Pareto optimal. From the generalized Nash solution, $\phi = \eta_0(p_K - q_0)k$. Hence, the capital producer's total surplus is $(1 - \eta_0)(p_K - q_0)k$.

The value of one unit of idle capital, q_0, is the solution to:

$$rq_0 = \alpha_0(1 - \eta_0)(p_K - q_0) + \dot{q}_0. \tag{103}$$

The opportunity cost of holding a unit of idle capital is given by the left side of (103). According to the right side, the unit of capital is sold to a broker at rate α_0 at which point the capital producer makes a profit equal to $(1 - \eta_0)(p_K - q_0)$. Capital producers choose the level of investment to maximize $i_t(q_0 - 1)$. Hence, in any equilibrium with $i_t > 0$, $q_0 = 1$. It follows from (103) that the price of capital solves

$$p_K = 1 + \frac{r}{\alpha_0(1 - \eta_0)}. \tag{104}$$

It is equal to its production cost, one, plus the holding cost until this unit can be sold. The bargaining power of the broker gives rise to a multiplier, $1/(1-\eta_0)$, of the holding cost of idle capital, r/α_0. The logic is as follows. The profit that the capital producer makes by selling one unit of capital, $(1 - \eta_0)(p_K - 1)$, must cover the holding cost, r/α_0. As the holding cost increases by one, the price of capital, p_K, must increases but by more than one because brokers captures a fraction of the gains that the sale of the capital goods generates.

The value of a unit of capital employed by a firm of type $j \in \{L, H\}$, q_j, solves an equation analogous to (11) where the price of capital, p_K, is now given by (104). Using the same steps as before, the Tobin's q is:

$$q_j = \frac{[r + \alpha(1 - \eta)]\, A_j + \lambda \bar{A}}{[r + \alpha(1 - \eta)]\, [r + \lambda + \alpha(1 - \eta)]} + \frac{\alpha(1 - \eta)p_K}{r + \alpha(1 - \eta)} \quad \forall j \in \{L, H\}. \tag{105}$$

Each firm chooses its capital stock to maximize $(q_j - p_K)k$. Hence, in order for the capital market to clear, $q_L < p_K$ and

$$q_H = p_K. \tag{106}$$

An equilibrium can be reduced to a 4-tuple, $(p_K, q_L, q_H, r) \in \mathbb{R}_+ \times [0, 1] \times \{1\} \times (\rho, +\infty)$, solution to (104), (105), (106) and $r < \rho/(1 - \theta)$ if $\theta < 1$. I substitute p_K by its expression given by (104) into (105) for $j = H$ to obtain the following proposition.

Proposition 12 ***(Real interest rate with frictional primary capital market.)*** *Assume*

$$\rho \left[1 + \frac{\rho}{\alpha_0(1 - \eta_0)}\right] < \frac{[\rho + \alpha(1 - \eta)]\, A_H + \lambda \bar{A}}{\rho + \lambda + \alpha(1 - \eta)}. \tag{107}$$

Moreover, $\theta \geq 1$ *or, if* $\theta < 1$,

$$\frac{\rho}{(1-\theta)}\left[1+\frac{\rho}{\alpha_0(1-\eta_0)(1-\theta)}\right] > \frac{\left[\rho+\alpha(1-\eta)(1-\theta)\right]A_H+\lambda\bar{A}}{\rho+(1-\theta)\left[\lambda+\alpha(1-\eta)\right]}. \tag{108}$$

The equilibrium real interest rate along a balanced-growth path with $g > 0$ *is uniquely determined as the solution to*

$$r\left[1+\frac{r}{\alpha_0(1-\eta_0)}\right] = \frac{\left[r+\alpha(1-\eta)\right]A_H+\lambda\bar{A}}{r+\lambda+\alpha(1-\eta)}. \tag{109}$$

It is such that $r \in (\rho, A_H)$, $\partial r/\partial\alpha_0 > 0$, *and* $\partial r/\partial\eta_0 < 0$.

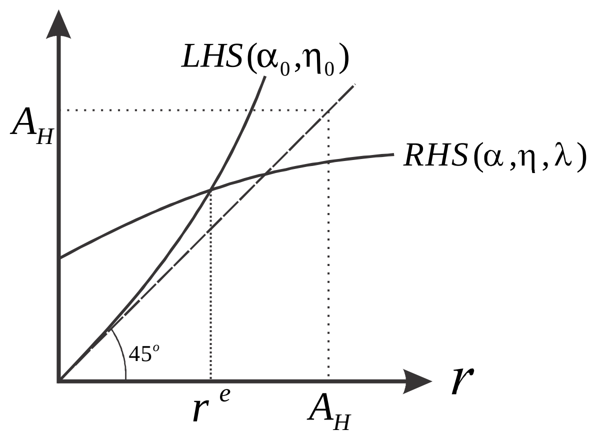

Figure 13: Determination of the real interest rate when the primary and secondary capital markets are frictional.

Equation (109), which is represented graphically in Figure 13, determines a unique $r > 0$. The right side depends on the frictions in the secondary market, α and η, while the left side depends on the frictions in the primary market, α_0 and η_0. If α_0 increases or η_0 decreases, then the left side falls and r increases. If α increases or η decreases, then the right side increases and r increases. Under condition (107), r is larger than ρ, which guarantees that the growth rate of the economy, $g = (r-\rho)/\theta$, is positive. For (107) to be satisfied, the term $\alpha_0(1-\eta_0)$ must be sufficiently large. Equivalently, the frictions in the primary market cannot be too large.

Capital misallocation

As in the previous section, there is dynamic misallocation in that the rate of capital accumulation is inefficiently low and there is static misallocation in that a fraction of the capital is not operated by type-H firms.

Dynamic misallocation In order to understand the sources of the dynamic misallocation, it can be instructive to rewrite (109) as follows:

$$r = \left[A_H - \overbrace{\frac{\lambda \pi_L (A_H - A_L)}{r + \lambda + \alpha(1-\eta)}}^{\text{misallocation discount}} \right] \overbrace{\left[1 + \frac{r}{\alpha_0(1-\eta_0)} \right]^{-1}}^{\text{primary markdown}}.$$

It shows the different forces that push the real interest rate below its first-best value, A_H. There is the misallocation discount described earlier according to which the real interest rate falls below its frictionless value in the presence of idiosyncratic productivity shocks due to the frictions to reallocate capital. The novelty is the second term on the right side, labelled *primary markdown*, which corresponds to the markdown of the wholesale price of newly-produced capital goods that results from the market power of primary brokers.

I now investigate how the reduction of the frictions on each side of the market affects misallocation. Consider the limit as the frictions in primary market vanish. Since $\eta_0 < 1$, then as $\alpha_0 \to +\infty$, r approaches to the solution given by (24). Graphically, the left side approaches the 45$^{\text{o}}$ line. Suppose next that the frictions in the secondary market vanish, $\alpha(1-\eta) \to +\infty$, while the primary market remains frictional. Then r converges to the solution to

$$r \left[1 + \frac{r}{\alpha_0(1-\eta_0)} \right] = A_H.$$

Even though all units of employed capital are allocated to firms of type H, there is some idle or unemployed capital. As a result, the real interest rate is below A_H.

Static misallocation I turn to the allocation of the capital stock. In a first-best allocation, all the capital should be held by type-H firms. In the presence of frictions, some capital will be idle and some capital will be held by type-L firms. Let K_I denote the aggregate stock of idle capital, that is the newly produced capital that has not reached the market and is not employed by any firm. It obeys the following law of motion:

$$\dot{K}_{I,t} = \dot{K}_t - \alpha_0 K_{I,t}. \tag{110}$$

The stock of idle capital increases with investment, $\dot{K}_t$, but it decreases with the flow of idle capital that is sold to brokers and allocated to firms at rate α_0. In the following proposition, I denote $\hat{k}_j \equiv K_j/K$ the share of total capital in state $j \in \{I, L, H\}$.

Proposition 13 *On a balanced-growth path, the distribution of the capital stock across states is:*

$$\hat{k}_I = \frac{g}{g+\alpha_0} \tag{111}$$

$$\hat{k}_L = \frac{\lambda \pi_L \alpha_0}{(g+\lambda+\alpha)(g+\alpha_0)} \tag{112}$$

$$\hat{k}_H = \frac{\alpha_0 (g+\alpha+\lambda\pi_H)}{(g+\alpha+\lambda)(g+\alpha_0)}. \tag{113}$$

The unemployment rate of the capital, $K_{I,t}/K_t$, increases with the growth rate of the economy but decreases with the meeting rate in the primary capital market, α_0. The share of capital employed by firms of type L, $\hat{k}_L$, increases with α_0 and decreases with g. The share of capital held by the most productive firms, $\hat{k}_H = K_H/K$, increases with α_0 while the effect of g is ambiguous. In order to measure the static misallocation, I express the aggregate output as

$$Y_t = K_t \sum_{j\in\{L,H\}} \hat{k}_j A_j = K_t \left(\frac{\alpha_0}{g+\alpha_0}\right)\left[A_H - \frac{\lambda\pi_L(A_H - A_L)}{\lambda+\alpha+g}\right]. \tag{114}$$

The TFP is equal to the one in (19) scaled down be the employment rate of the capital stock, $\alpha_0/(g+\alpha_0)$. While an increase in g reduces the misallocation across the employed capital, it also reduces the employment rate of capital.

Proof of Proposition 12. From (105) for $j = H$ and using that $q_H = p_K$,

$$rp_K = \frac{[r+\alpha(1-\eta)]\,A_H + \lambda\bar{A}}{r+\lambda+\alpha(1-\eta)}.$$

Substituting p_K by its expression given by (104) one obtain (109). The left side is

$$LHS(r) \equiv r\left[1+\frac{r}{\alpha_0(1-\eta_0)}\right].$$

It is a quadratic function with two roots, $r = 0$ and $r = -\alpha_0(1-\eta_0)$. It is increasing and convex for all $r > 0$. The right side can be rewritten as

$$RHS(r) \equiv A_H - \frac{\lambda(A_H - \bar{A})}{r+\lambda+\alpha(1-\eta)}.$$

It is increasing, concave, strictly positive at $r = 0$, and it approaches A_H as $r \to +\infty$. Hence, as depicted in Figure 13, there is a unique $r \in (0, A_H)$ solution to $LHS(r) = RHS(r)$. The real interest rate is larger than ρ if and only if $RHS(\rho) > LHS(\rho)$, i.e., (107) holds. The condition $r > g$ can be reexpressed as $r(1-\theta) < \rho$. It holds if $\theta \geq 1$. Suppose $\theta < 1$.

Then $r - g > 0$ can be rewritten as $r < \rho/(1-\theta)$. It holds if and only if $LHS\left[\rho/(1-\theta)\right] > RHS\left[\rho/(1-\theta)\right]$, i.e., (108) holds. ■

Proof of Proposition 13.

I focus on a balanced growth path where K_I and K grow at the same rate, g. Hence, $\dot{K}_{I,t} = gK_{I,t}$ and $\dot{K}_t = gK_t$ so that (110) can be rewritten as

$$K_{I,t} = \frac{g}{g + \alpha_0} K_t, \tag{115}$$

which gives (111). The capital held by type-H firms evolves according to

$$\dot{K}_H = \alpha(K - K_I - K_H) + \lambda\pi_H(K - K_I - K_H) - \lambda\pi_L K_H + \alpha_0 K_{I,t}. \tag{116}$$

The first term on the right side of (116) corresponds to the stock of capital that is employed by firms of type L, $K_L = K - K_I - K_H$, and that reaches the secondary capital market at rate α to be reallocated to type-H firms. The second term corresponds to the capital employed by firms of type L who receive a positive productivity shock with probability π_H so that the capital is now owned by firms of type H. The third term corresponds to firms of type H that receive a negative productivity shock at rate $\lambda(1 - \pi_H)$. The last term on the right side corresponds to the unemployed capital that reaches the market at rate α_0 and is allocated to type-H firms. Using that $\dot{K}_H = gK_H$ and substituting K_I by its expression, $K_H/K = \hat{k}_H$ given by (113). The capital held by type-L firms evolves according to

$$\dot{K}_L = \lambda\pi_L(K - K_I - K_L) - \lambda\pi_H K_L - \alpha K_L. \tag{117}$$

The interpretation is similar to the one of (116). Using that $\dot{K}_L = gK_L$, $\hat{k}_L \equiv K_L/K$ is given by (112). ■

C5. Endogenous bargaining power of brokers

Suppose that brokers can exert some costly effort to increase their bargaining power, η. The cost of this effort is $\psi(\eta)K$, where $\psi(\eta) = \bar{\psi}\eta$ with $\bar{\psi} > 0$. It is scaled by the aggregate capital stock to obtain a balanced growth path. In practice, we can interpret the investment in bargaining strength as follows: brokers improve their pricing strategy and marketing techniques and acquire information about firms to better price-discriminate.

The problem of the intermediary in (13) is generalized as follows:

$$rw = \max_{\eta\in[0,1]} \left\{ -\psi(\eta)K + \alpha\eta(1-q_L) \int k d\Upsilon(L,k) \right\} + \dot{w}. \tag{118}$$

On the right side of (118), the intermediary chooses its bargaining power, η, at some cost, $\psi(\eta)K$. The expected profits of the intermediary are linear in its bargaining power. Using that $\int k d\Upsilon(L,k) = K_L$, the first-order condition is

$$\eta \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \quad \text{if } \bar{\psi} \begin{cases} < \\ = \\ > \end{cases} \alpha(1-q_L)\hat{k}_L. \tag{119}$$

When deciding to invest in its bargaining power, the intermediary compares the cost $\bar{\psi}$ to the expected return $\alpha(1-q_L)\hat{k}_L$. It chooses to invest in a higher bargaining power when the share of capital that is misallocated increases, when trading frictions get smaller, and when the Tobin's q of low-productivity firms decreases.

We focus on symmetric Nash equilibria where all brokers choose the same η. Substitute $\hat{k}_L$ and q_L by their expressions to rewrite the first-order condition as:

$$\eta \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \quad \text{if } \bar{\psi} \begin{cases} < \\ = \\ > \end{cases} \frac{\theta\alpha\lambda\pi_L\left(A_H - A_L\right)}{\left[r-\rho+\theta\left(\alpha+\lambda\right)\right]\left[r+\lambda+\alpha(1-\eta)\right]}, \tag{120}$$

where r is a decreasing function of η given by (24). The term on the right is increasing in η. Hence, there is at most one average value of η that makes individual brokers indifferent between investing in their bargaining strength or not. It follows that the best-response function given by (120) is a weakly increasing step function.

Proposition 14 ***(Endogenous rent seeking.)*** *Suppose* $\theta \geq 1$ *and*

$$\rho < \frac{\rho A_H + \lambda\bar{A}}{\rho+\lambda}. \tag{121}$$

There exist $0 < \bar{\psi}_0 < \bar{\psi}_1 < +\infty$ *such that the following is true:*

1. For all $\bar{\psi} \geq \bar{\psi}_0$, there exists an equilibrium with $\eta = 0$ and $r = r_0 > \rho$ solution to

$$r_0 = \frac{(r_0 + \alpha) A_H + \lambda \bar{A}}{r_0 + \lambda + \alpha}. \tag{122}$$

2. For all $\bar{\psi} \leq \bar{\psi}_1$, there exists an equilibrium with $\eta = 1$ and $r = r_1 > \rho$ solution to

$$r_1 = \frac{r_1 A_H + \lambda \bar{A}}{r_1 + \lambda}. \tag{123}$$

3. For all $\bar{\psi} \in (\bar{\psi}_0, \bar{\psi}_1)$, there is an interior equilibrium, $(r, \eta) \in \mathbb{R}^+ \times (0,1)$, solution to

$$\bar{\psi} = \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r - \rho + \theta(\alpha + \lambda)][r + \lambda + \alpha(1-\eta)]} \tag{124}$$

$$r = \frac{[r + \alpha(1-\eta)] A_H + \lambda \bar{A}}{r + \lambda + \alpha(1-\eta)}. \tag{125}$$

4. As $\alpha \to +\infty$, the length of the interval, $(\bar{\psi}_0, \bar{\psi}_1)$, is maximum with

$$\lim_{\alpha \to +\infty} \bar{\psi}_0(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to +\infty} \bar{\psi}_1(\alpha) = \frac{\lambda \pi_L (A_H - A_L)}{r_1 + \lambda}.$$

Proof. Part 1. From (120), an equilibrium without rent seeking, $\eta = 0$, if and only if

$$\bar{\psi} \geq \bar{\psi}_0 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_0 - \rho + \theta(\alpha + \lambda)](r_0 + \lambda + \alpha)},$$

where $r_0(\alpha)$ is the solution to (122). Note that $\bar{\psi}_0$ is a continuous function of α that is equal to 0 when $\alpha = 0$ and $\alpha = +\infty$ and is positive for all $\alpha \in (0, +\infty)$.

Part 2. From (120), an equilibrium with $\eta = 1$ exists if and only if

$$\bar{\psi} \leq \bar{\psi}_1 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_1 - \rho + \theta(\alpha + \lambda)](r_1 + \lambda)},$$

where r_1 solves (123). From (121), $r_1 > \rho$. Note that r_1 is independent of α. Moreover, $\bar{\psi}_1$ is an increasing function of α that is equal to 0 when $\alpha = 0$ and $\lambda \pi_L (A_H - A_L) / (r_1 + \lambda)$ when $\alpha = +\infty$.

Part 3. Using that $r_0 > r_1$,

$$\bar{\psi}_0 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_0 - \rho + \theta(\alpha + \lambda)](r_0 + \lambda + \alpha)} < \bar{\psi}_1 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_1 - \rho + \theta(\alpha + \lambda)](r_1 + \lambda)}.$$

It can be checked that for all $\bar{\psi} \in (\bar{\psi}_0, \bar{\psi}_1)$, there is an interior equilibrium, $(r, \eta) \in \mathbb{R}^+ \times (0,1)$, solution to (124)-(125). Equation (124) gives a positive relation between r and η while (125) gives a negative relation between r and η. The solution is interior if r solution to (124) when

$\eta = 1$ is larger than r_1, i.e., $\bar{\psi} < \bar{\psi}_1$, and if r solution to (124) when $\eta = 0$ is less than r_0, i.e., $\bar{\psi} > \bar{\psi}_0$.

Part 4. As $\alpha \to +\infty$,

$$\bar{\psi}_0 \equiv \frac{\theta\lambda\pi_L (A_H - A_L)}{[r_0 - \rho + \theta(\alpha + \lambda)] [1 + (r_0 + \lambda)/\alpha]} \to 0,$$

and hence $\lim_{\alpha \to +\infty} \bar{\psi}_0(\alpha) = 0$. From Part 2, $\bar{\psi}_1$ is an increasing function of α that approaches $\lambda\pi_L (A_H - A_L) / (r_1 + \lambda)$ as $\alpha \to +\infty$. ■

The first part of Proposition 14 shows an equilibrium without rent seeking by brokers, $\eta = 0$, exists if the marginal cost of rent seeking is larger than some threshold, $\bar{\psi}_0$. This threshold is a nonmonotone function of α which approaches 0 as either $\alpha \to 0$ or $\alpha \to +\infty$. It is maximum for some intermediate value of α. So, when $\bar{\psi}$ is not too large, an equilibrium where brokers have no market power is more likely when the market is either very illiquid or very liquid.

The second part of Proposition 14 shows an equilibrium with maximum rent seeking, $\eta = 1$, exists if the marginal cost of rent seeking is lower than some threshold, $\bar{\psi}_1$. This threshold is an increasing function of α that is equal to 0 when $\alpha = 0$ and that is positive as $\alpha \to +\infty$. So, an equilibrium with monopolist brokers is more likely in liquid markets.

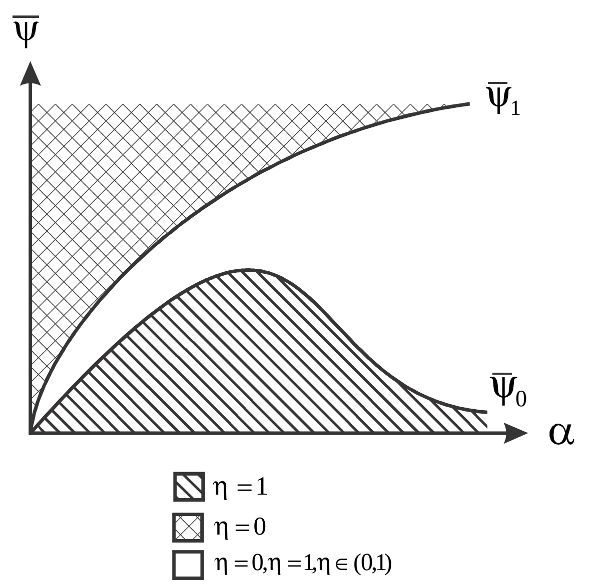

Figure 14: Typology of balanced-growth equilibria with endogenous rent seeking

The threshold $\bar{\psi}_0$ is lower than $\bar{\psi}_1$. Hence, when $\bar{\psi}$ is between $\bar{\psi}_0$ and $\bar{\psi}_1$, there are multiple equilibria. There is a high-growth equilibrium where brokers do not invest in their bargaining power and there is a low-growth equilibrium where brokers have maximum bargaining power. The share of capital that is misallocated is the same in both equilibria.

However, the competitive equilibrium has a higher real interest rate and a higher growth rate. There is also an intermediate equilibrium where $\eta \in (0,1)$. From (124)-(125), the real interest rate solves

$$\bar{\psi} = \frac{\theta\alpha\left(A_H - r\right)}{r - \rho + \theta\left(\alpha + \lambda\right)}.$$

It increases in α and decreases in $\bar{\psi}$. The typology of equilibria is represented in Figure 14 in the parameter space $(\alpha, \bar{\psi})$.

We now consider equilibria at the frictionless limit where $\alpha \to +\infty$. The threshold $\bar{\psi}_0$ goes to 0 so that an equilibrium with $\eta = 0$ always exists at the frictionless limit. The threshold below which $\eta = 1$ is such that

$$\lim_{\alpha \to +\infty} \bar{\psi}_1 \equiv \frac{\lambda \pi_L \left(A_H - A_L\right)}{r_1 + \lambda} > 0.$$

So there can also exist equilibria with monopolist brokers when trading frictions vanish. We now turn to interior equilibria and denote $\hat{\alpha}_\infty = \lim_{\alpha \to +\infty} \alpha(1-\eta)$. From (124) and (125),

$$\begin{aligned} \bar{\psi} &= \frac{\lambda \pi_L \left(A_H - A_L\right)}{A_H - \bar{\psi} + \lambda + \hat{\alpha}_\infty}. \\ r &= A_H - \bar{\psi}. \end{aligned}$$

So, as the limit, $\eta \to 1$ but the effective meeting rate of firms, $\alpha(1-\eta)$, remains bounded. The real interest rate is below the one of a frictionless economy, A_H.

C6. When the knowledge externality does not sustain growth

I now describe the case where the knowledge externality is not sufficiently strong to sustain long-run growth. I will show that the model can still be useful to explain differences in per capita GDP due to differences in market structures.

The technology of a firm takes the form, $A_j K^\zeta k^\beta$, where $\zeta + \beta \in (0,1)$. Thus, there are decreasing returns from scaling both K and k up. In a steady-state equilibrium, the capital stock, K_t, is constant, the growth rate is $g = 0$, and, from the Euler equation, the real interest rate is $r = \rho$.

By the same reasoning as in Lemma 2, the value function of the firm is

$$v_j(k) = \frac{\hat{A}_j K^\zeta k^\beta + \alpha(1-\eta)k}{\rho + \alpha(1-\eta)} + \Lambda_j.$$

The optimal choice of capital maximizes $v_j(k) - k$, i.e.,

$$k_j = \left(\frac{\beta \hat{A}_j}{\rho}\right)^{\frac{1}{1-\beta}} K^{\frac{\zeta}{1-\beta}}. \tag{126}$$

By market clearing, $\sum_{j=1}^J \pi_j k_j = K$. The right side is the supply of capital while the left side is the demand from firms accessing the market. Substituting k_j by its expression given by (126), one obtains the aggregate capital stock in a steady-state equilibrium:

$$K = \left(\frac{\beta \hat{r}^*}{\rho}\right)^{\frac{1}{1-\beta-\zeta}} \quad \text{where} \quad \hat{r}^* \equiv \left(\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

An increase in market frictions, through a reduction of α or an increase in η, reduces $\hat{r}^*$, which then reduces K. The effect is magnified by the learning-by-investing externality since $\partial \ln(K)/\partial \ln(\hat{r}^*) = (1-\beta-\zeta)^{-1} > 1$ increases with ζ.

Total output is the sum of the output flows across all firms,

$$Y \equiv \sum_{(i,j)\in\{1,\dots,J\}^2} \int_0^{+\infty} A_j K^\zeta k_i^\beta \gamma(i,j,\tau) d\tau. \tag{127}$$

Lemma 10 *Suppose α and λ are large relative to ρ and $\eta = 0$. Aggregate output is approximately equal to*

$$Y = \left(\frac{\beta}{\rho}\right)^{\frac{\zeta+\beta}{1-\beta-\zeta}} \hat{r}^{*\frac{1}{1-\beta-\zeta}}. \tag{128}$$

Proof. Substitute $\gamma(i,j,\tau)$ by its expression given by (38) into (127) and simplify to obtain:

$$Y = K^{\zeta} \sum_{i\in\{1,\dots,J\}} \pi_i \left(\frac{\alpha A_i + \lambda \bar{A}}{\alpha+\lambda}\right) k_i^{\beta}.$$

After substituting k_i by its expression given by (126),

$$Y = \left(\frac{\beta}{\rho}\right)^{\frac{\zeta+\beta}{1-\beta-\zeta}} \hat{r}^{*\frac{\zeta}{(1-\beta-\zeta)(1-\beta)}} \sum_{i\in\{1,\dots,J\}} \pi_i \left(\frac{\alpha A_i + \lambda \bar{A}}{\alpha+\lambda}\right) \hat{A}_i^{\frac{\beta}{1-\beta}}.$$

Under the assumption that α and λ are large relative to ρ and $\eta = 0$,

$$\hat{A}_i \equiv \frac{\left[\rho + \alpha(1-\eta)\right] A_i + \lambda \bar{A}}{\rho + \alpha(1-\eta) + \lambda}$$

can be approximated by $\hat{A}_i \approx \left(\alpha A_i + \lambda \bar{A}\right)/(\alpha+\lambda)$. Substitute into the expression for Y to obtain (128). ■

From (128),

$$\frac{\partial \ln Y}{\partial \alpha} = (1-\beta-\zeta)^{-1} \frac{\partial \ln \hat{r}^*}{\partial \alpha}.$$

Market frictions affect output through $\hat{r}^*$. The effect is amplified by the knowledge externality that can make the elasticity of Y with respect to $\hat{r}^*$ arbitrarily large.

C7. Externalities and growth

I describe a version of the model where firms invest in some input, h, that generates the externality sustaining economy growth. The input, h, can be any type of intangible or human capital that contributes to the formation of nonrival knowledge. The objective is twofold. First, the logic of the model in the main text goes through even if the knowledge externality does not operates through k. Second, the choice of h by firms will respond endogenously to the mismatch between their idiosyncratic productivity and their capital stock.

The technology takes the form:

$$f_j(h,k) = z_j H^{(1-\nu)(1-\beta)} h^{\nu} k^{(1-\nu)\beta}, \tag{129}$$

where $h \in \mathbb{R}^+$ represents the investment in knowledge to operate the capital stock and $z_j \in \mathbb{R}^+$ is idiosyncratic productivity. The cost of this knowledge in terms of the consumption good is h. For simplicity, I treat h as a control variable. The external effect that generates growth operates through the aggregate knowledge across all firms, $H \equiv \int_0^1 h_x dx$.

At each point in time, the firm chooses $h \in \mathbb{R}^+$ to maximize $f_j(h,k) - h$. The optimal choice is

$$h_{j,t}(k) = (\nu z_j)^{\frac{1}{1-\nu}} H_t^{1-\beta} k^{\beta}. \tag{130}$$

So, h increases with the capital stock, k, and with the idiosyncratic productivity, z_j. Substituting h by its expression given by (130) into (129), the net output of the firm is

$$y_{j,t}(k) \equiv f_j\left[h_{j,t}(k), k\right] - h_{j,t}(k) = A_j H_t^{1-\beta} k^{\beta}, \tag{131}$$

where

$$A_j \equiv (z_j \nu)^{\frac{1}{1-\nu}} \left(\frac{1-\nu}{\nu}\right).$$

The net output of the firm, once maximized with respect to knowledge acquisition, takes a similar form as the one studied earlier where K is replaced with H. From (130) and (131),

$$y_j(k) = \left(\frac{1-\nu}{\nu}\right) h_j(k). \tag{132}$$

So, the output of the firm is proportional to the investment in knowledge.

The aggregate knowledge is

$$H_t = \int h_j(k_{i,t-\tau}) d\Upsilon(i,j,\tau),$$

where $\Upsilon(i,j,\tau)$ is the distribution of firms' types, where τ is the time that has elapsed since the last access to the market, i is productivity at the last access to the market, j is current productivity. Substituting $h_j(k)$ by its expression given by (130), H solves

$$H_t = \left(\frac{\nu}{1-\nu}\right) \int A_j H_t^{1-\beta} (k_{i,t-\tau})^\beta d\Upsilon(i,j,\tau).$$

Using that $k_{i,t-\tau} = e^{-g\tau} k_{i,t}$, one can solve for H_t to obtain

$$H_t = \left(\frac{\nu}{1-\nu}\right)^{\frac{1}{\beta}} \left[\int e^{-\beta g\tau} A_j (k_{i,t})^\beta d\Upsilon(i,j,\tau)\right]^{\frac{1}{\beta}}. \tag{133}$$

I now turn to the determination of $k_{j,t}$, the optimal choice of capital of the firm when it accesses the market. It solves $\max_k \{\upsilon_j(k) - k\}$ where

$$\upsilon_{j,t}(k) = \frac{\hat{A}_j H_t^{1-\beta} k^\beta}{r - (1-\beta)g + \alpha(1-\eta)} + \frac{\alpha(1-\eta)}{\alpha(1-\eta)+r} k + \Lambda_{j,t}. \tag{134}$$

The solution is given by the same expression as in the main text where K_t is replaced with H_t:

$$\tilde{k}_j \equiv \frac{k_{j,t}}{H_t} = \left\{\frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]} \hat{A}_j\right\}^{\frac{1}{1-\beta}}. \tag{135}$$

I substitute $k_{i,t} = \tilde{k}_i H_t$ in (133) to obtain

$$1 = \left(\frac{\nu}{1-\nu}\right) \int e^{-\beta g\tau} A_j \tilde{k}_i^{\beta} d\Upsilon(i,j,\tau). \tag{136}$$

Using the observation that $g = (r-\rho)/\theta$ and $\tilde{k}_i$ are functions of r, (136) determines the real interest rate. In the following Lemma, I simplify the equilibrium condition, (136).

Lemma 11 *An equilibrium is a $r > \rho$ solution to*

$$\sum_{j=1}^{J} \pi_j \left[\frac{(\alpha+\beta g)A_j + \lambda \bar{A}}{\alpha+\lambda+\beta g}\right] \tilde{k}_j^{\beta} = \left(\frac{\alpha+\beta g}{\alpha}\right)\left(\frac{1-\nu}{\nu}\right), \tag{137}$$

such that $r(\theta-1)+\rho > 0$, where $g = (r-\rho)/\theta$ and $\tilde{k}_j$ is given by (135).

By substituting $\tilde{k}_j$ by its expression given by (135), the equation for r, (137), can also be reexpressed as

$$r = \left(\frac{\alpha}{\alpha+\beta g}\right)^{\frac{1-\beta}{\beta}} \left(\frac{\nu}{1-\nu}\right)^{\frac{1-\beta}{\beta}} \frac{\beta\left[\alpha(1-\eta)+r\right]}{r-(1-\beta)g+\alpha(1-\eta)} \left\{\sum_{j=1}^{J} \pi_j \left[\frac{(\alpha+\beta g)A_j + \lambda\bar{A}}{\alpha+\lambda+\beta g}\right] \hat{A}_j^{\frac{\beta}{1-\beta}}\right\}^{\frac{1-\beta}{\beta}}.$$

In the equilibrium condition (137), the real interest rate appears in the growth rate, g, and in the firms' demand for capital, $\tilde{k}_j$. The next proposition provides sufficient conditions for the existence of an equilibrium.

Proposition 15 *Assume $\theta > 1$. If*

$$\sum_{j=1}^{J} \pi_j \left(\frac{\alpha A_j + \lambda \bar{A}}{\alpha + \lambda} \right) \left\{ \frac{\beta}{\rho} \frac{\left[\rho + \alpha(1-\eta)\right] A_j + \lambda \bar{A}}{\rho + \alpha(1-\eta) + \lambda} \right\}^{\frac{\beta}{1-\beta}} > \frac{1-\nu}{\nu}, \tag{138}$$

then an equilibrium exists.

I now turn to the other equilibrium quantities. Using (132), aggregate output is related to the stock of knowledge as follows:

$$Y_t = \frac{1-\nu}{\nu} H_t.$$

Using the market clearing condition according to which $\alpha \sum_{j=1}^{J} \pi_j k_{j,t} = (\alpha + g) K_t$, and from (135), the relationship between H_t and K_t is given by

$$\frac{K_t}{H_t} = \left(\frac{\alpha}{\alpha + g} \right) \left\{ \frac{\beta \left[\alpha(1-\eta) + r\right]}{r \left[r - (1-\beta) g + \alpha(1-\eta)\right]} \right\}^{\frac{1}{1-\beta}} \sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}.$$

Combining the two equations above,

$$\begin{aligned} Y_t &= \left(\frac{\alpha + g}{\alpha} \right) \left(\frac{\alpha}{\alpha + \beta g} \right)^{\frac{1}{\beta}} \left(\frac{\nu}{1-\nu} \right)^{\frac{1-\beta}{\beta}} \\ &\quad \times \left\{ \sum_{j=1}^{J} \pi_j \left[\frac{(\alpha + \beta g) A_j + \lambda \bar{A}}{\alpha + \lambda + \beta g} \right] \hat{A}_j^{\frac{\beta}{1-\beta}} \right\}^{\frac{1}{\beta}} \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{-1} K_t. \end{aligned}$$

In Figure 15, I plot the real interest rate as a function of α and η. The parameter values are the same as the ones in the main text. In addition, I set $\nu = 0.85$. In accordance with the results from the main text, r can vary in a non-monotonic fashion with frictions. For instance, an increase in α can reduce the real interest rate and the growth rate of the economy. Conversely, an increase in η can raise r and g.

Proof of Lemma 11. Denote $\mathcal{I} \equiv \int e^{-\beta g \tau} A_j \tilde{k}_i^{\beta} d\Upsilon(i,j,\tau)$. The density of $\Upsilon(i,j,\tau)$ is

$$\gamma(i,j,\tau) = \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau}) \pi_j \right],$$

to obtain:

$$\begin{aligned} \mathcal{I} &= \sum_{(i,j) \in \{1,..,J\}^2} \int_0^{+\infty} e^{-\beta g \tau} A_j \tilde{k}_i^{\beta} \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau}) \pi_j \right] d\tau \\ &= \sum_{i,j} A_j \tilde{k}_i^{\beta} \pi_i \int_0^{+\infty} \left[\alpha e^{-(\alpha+\lambda+\beta g)\tau} \mathbb{I}_{\{i=j\}} + \alpha e^{-(\alpha+\beta g)\tau} (1 - e^{-\lambda\tau}) \pi_j \right] d\tau, \end{aligned}$$

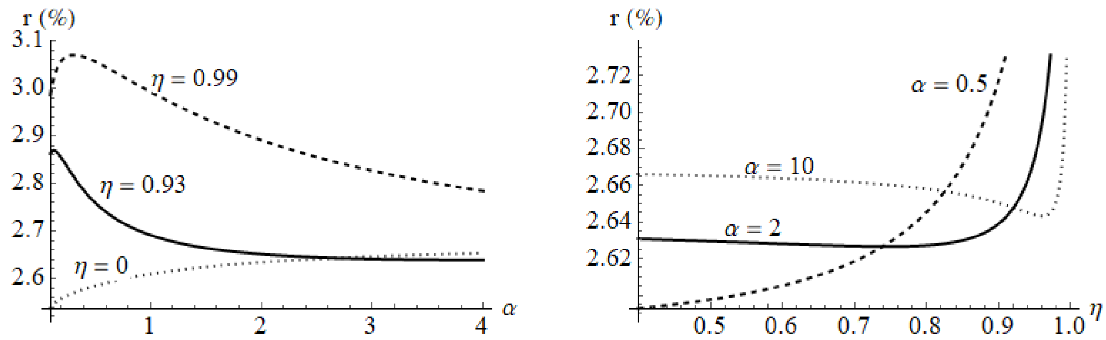

Figure 15: The relation between r and market frictions.

where the second equality is obtained by taking $A_j \tilde{k}_i{}^{\beta} \pi_i$ outside of the integral. Next, I compute the integral:

$$\mathcal{I} = \sum_{i,j} A_j \tilde{k}_i{}^{\beta} \pi_i \left[\frac{\alpha \mathbb{I}_{\{i=j\}}}{\alpha + \lambda + \beta g} + \frac{\alpha \lambda \pi_j}{(\alpha + \beta g)(\alpha + \lambda + \beta g)} \right].$$

I take the sum over the different values for j:

$$\mathcal{I} = \frac{\alpha}{\alpha + \beta g} \sum_i \pi_i \left[\frac{(\alpha + \beta g) A_i + \lambda \bar{A}}{\alpha + \lambda + \beta g} \right] \tilde{k}_i{}^{\beta}.$$

where I used that $\bar{A} = \sum_j \pi_j A_j$. I plug this expression for $\mathcal{I}$ into (136) to obtain (137). ■

Proof of Proposition 15. From (137), an equilibrium is a $r > \rho$ solution to $\Gamma(r) = 0$ where

$$\Gamma(r) \equiv \sum_j \pi_j \left[\frac{[\alpha\theta + \beta(r - \rho)] A_j + \theta \lambda \bar{A}}{\theta(\alpha + \lambda) + \beta(r - \rho)} \right] \tilde{k}_j{}^{\beta} - \left(\frac{\alpha\theta + \beta(r - \rho)}{\alpha\theta} \right) \left(\frac{1 - \nu}{\nu} \right),$$

where

$$\tilde{k}_j^{\beta} = \left\{ \frac{\beta [\alpha(1 - \eta) + r]}{r [r - (1 - \beta) g + \alpha(1 - \eta)]} \hat{A}_j \right\}^{\frac{\beta}{1-\beta}}$$

and

$$\hat{A}_j = \frac{[r - (1 - \beta)(r - \rho)/\theta + \alpha(1 - \eta)] A_j + \lambda \bar{A}}{r - (1 - \beta)(r - \rho)/\theta + \alpha(1 - \eta) + \lambda}.$$

Since $\theta > 1$,

$$r - (1 - \beta) \left(\frac{r - \rho}{\theta} \right) = \frac{[\theta - (1 - \beta)] r + (1 - \beta)\rho}{\theta} > 0$$

for all $r \geq \rho$. Hence, the function Γ is continuous for all $r \geq \rho$. Using that

$$\lim_{r \to +\infty} \left[\frac{[\alpha\theta + \beta(r - \rho)] A_j + \theta \lambda \bar{A}}{\theta(\alpha + \lambda) + \beta(r - \rho)} \right] \tilde{k}_j^{\beta} = 0,$$

it follows that $\Gamma(+\infty) = -\infty$. Hence, for an equilibrium to exists it suffices that $\Gamma(\rho) > 0$, i.e.,

$$\Gamma(\rho) = \sum_j \pi_j \left(\frac{\alpha A_j + \lambda \bar{A}}{\alpha + \lambda} \right) \tilde{k}_j{}^\beta - \left(\frac{1-\nu}{\nu} \right) > 0$$

where

$$\tilde{k}_j^\beta = \left(\frac{\beta \hat{A}_j}{\rho} \right)^{\frac{\beta}{1-\beta}}$$

and

$$\hat{A}_j = \frac{\left[\rho + \alpha(1-\eta)\right] A_j + \lambda \bar{A}}{\rho + \alpha(1-\eta) + \lambda}.$$

■

C8. Embodied technological progress

I now assume that technological progress is embodied into the new units of capital. The output flow of a firm is $A_j k^\beta$. There is no aggregate productivity term that grows over time due to technological progress. However, the new units of capital are produced with smaller and smaller quantities of the consumption good over time. Denote φ_t this quantity which represents the technology of the capital producer. If $\dot{\varphi}_t < 0$, technology progress permits to produce equipment or machines that are more productive for the same amount of the consumption good. The problem of capital producers is $\max_{i \geq 0}\{(p_{K,t} - \varphi_t)\, i\}$. Hence, in any equilibrium with positive investment, the price of capital in the inter-broker market is $p_{K,t} = \varphi_t$.

The value of a firm solves

$$\begin{aligned} r\upsilon_{j,t}(k) &= A_j k^\beta + \lambda \sum_{x=1}^{J} \pi_x \left[\upsilon_{x,t}(k) - \upsilon_{j,t}(k)\right] \\ &\quad + \alpha(1-\eta) \max_{k' \geq 0} \left[\upsilon_{j,t}(k') - \varphi_t(k'-k) - \upsilon_{j,t}(k)\right] + \dot{\upsilon}_{j,t}(k), \end{aligned} \tag{139}$$

for $j \in \{1, ..., J\}$. It can be seen from the first term on the right side of (139) that the output flow of the firm is constant over time. Since technological progress is embodied into the capital, a firm does not benefit from it unless it adjusts its capital stock. The third term on the right side captures the opportunity of the firm to readjust its capital. It can be done at the price φ_t which embodies the technological advances in producing capital goods.

Following Jovanovic and Rousseau (2002), suppose that $\varphi_t = \varphi K_t^{-(1-\beta)}$. The term K_t captures the assumption of learning by doing. As the capital stock grows, capital producers learn how to produce capital at a lower cost over time. In an equilibrium where K_t grows at rate g, $\dot{\varphi}_t/\varphi_t = -(1-\beta)g$. It follows that the expected resale value of capital from the standpoint of a firm at time t who accesses the market infrequently is

$$\begin{aligned} \mathbb{E}\left[e^{-r(T-t)}\varphi_T\right] &= \mathbb{E}\left[e^{-r(T-t)}\varphi\left(K_t\right)^{-(1-\beta)} e^{-(1-\beta)g(T-t)}\right] \\ &= \frac{\alpha(1-\eta)}{r + \alpha(1-\eta) + (1-\beta)g}\varphi\left(K_t\right)^{-(1-\beta)}. \end{aligned} \tag{140}$$

The capital can be sold when the firm accesses the market at time T. The length of time, $T-t$, is exponentially distributed with mean $1/[\alpha(1-\eta)]$.

I now compute the cumulated output of the firm from t to T. It is equal to

$$F_j(k) = \mathbb{E}\left[\int_t^T e^{-r(s-t)} A_{x(s)} k^\beta dt\right], \tag{141}$$

where $x(t) \in \{1, ..., J\}$ is the productivity type at time t. It solves the following HJB equation:

$$[r + \alpha(1-\eta)] F_j(k) = A_j k^\beta + \lambda \sum_{x=1}^{J} \pi_x [F_x(k) - F_j(k)].$$

After some calculation,

$$F_j(k) = \frac{[r + \alpha(1-\eta)] A_j + \lambda \bar{A}}{[r + \alpha(1-\eta) + \lambda][r + \alpha(1-\eta)]} k^\beta. \tag{142}$$

It is a similar expression as in the main text except that it does not depend on the rate of technological progress.

The value of the firm is the sum of its cumulated output until its next effective access to the market, the expected resale value of the capital, and a continuation value term that does not depend on k, i.e.,

$$\upsilon_{j,t}(k) = \frac{\hat{A}_j k^\beta}{r + \alpha(1-\eta)} + \frac{\alpha(1-\eta)}{r + \alpha(1-\eta) + (1-\beta)g} \varphi (K_t)^{-(1-\beta)} k + \Lambda_{j,t}, \tag{143}$$

where effective productivity is equal to

$$\hat{A}_j \equiv \frac{[r + \alpha(1-\eta)] A_j + \lambda \bar{A}}{r + \alpha(1-\eta) + \lambda}. \tag{144}$$

The problem of the firm who accesses the market at time t is $\max \{\upsilon_{j,t}(k) - \varphi_t k\}$. The first-order condition is:

$$\left(\frac{k_{j,t}}{K_t}\right)^{1-\beta} = \left[1 + \frac{(1-\beta)g}{r + \alpha(1-\eta)}\right] \frac{\beta \hat{A}_j}{\varphi [r + (1-\beta)g]}. \tag{145}$$

If $\alpha(1-\eta) < +\infty$ and $g > 0$, the first term on the right side is strictly greater than one, which increases the demand for capital. Firms anticipate that the price of capital goods is falling over time and that they will want to acquire more capital in the future. In order to mitigate the associated holdup problem, they anticipate on their future demand and raise their capital immediately.

The market clearing condition is as before,

$$\alpha \sum_{j=1}^{J} \pi_j \left(\frac{k_{j,t}}{K_t}\right) = \alpha + g. \tag{146}$$

By substituting $(k_{j,t}/K_t)$ by its expression given by (145), it can be reexpressed as:

$$r + (1-\beta)g = \left(\frac{\alpha}{\alpha + g}\right)^{1-\beta} \left[1 + \frac{(1-\beta)g}{r + \alpha(1-\eta)}\right] \frac{\beta}{\varphi} \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}\right)^{1-\beta}. \tag{147}$$

The left side of (147) corresponds to the user cost of capital, composed of the real interest rate net of the rate of appreciation of the price of capital goods, in a frictionless economy. The right side takes into account the effects of market frictions. The first term is the congestion effect of new capital goods when only a flow of firms can participate in the market. The second term is the front-loading effect when firms anticipate that their demand for capital will grow due to the price of capital falling over time at rate $(1-\beta)g$. The last two terms correspond to the equilibrium user cost of capital in a frictionless economy where A_j has been replaced with $\hat{A}_j$. Finally, by the Euler equation of the household,

$$g = \frac{r-\rho}{\theta}.$$

Thus, (147) determines the equilibrium real interest rate.

I now show that r can be larger than its value in a frictionless economy. From (147), in the absence of market frictions, $\alpha(1-\eta) \to +\infty$, r solves

$$\frac{[\theta + (1-\beta)]\, r - (1-\beta)\rho}{\theta} = \frac{\beta}{\varphi}\left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

Consider next the economy with market frictions. Suppose $\eta = 1$, dealers have all the bargaining power, and take the limit as $\alpha \to +\infty$. The real interest rate solves

$$r + (1-\beta)g = \left[1 + \frac{(1-\beta)g}{r}\right]\frac{\beta}{\varphi}\left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

The term between squared brackets on the right side is greater than one if $g > 0$. Thus, if a solution for r exists, it is greater than the frictionless value.

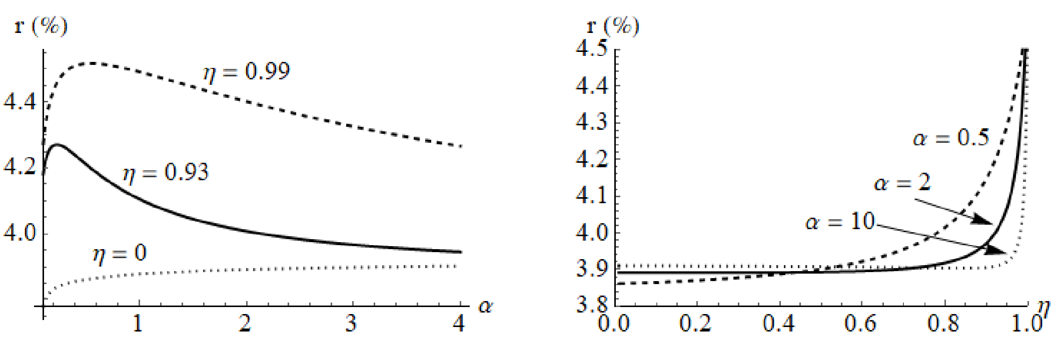

Figure 16: Real interest rate and market frictions under embodied technological progress. Parameter values: $\rho = 0.01$, $\theta = 2$, $\beta = 0.4$, $\lambda = 0.5$, $A_L = 0.089$, $A_H = 0.146$, $\pi_H = 0.5$, $\varphi = 1$.

References

[1] Jovanovic, Boyan, and Peter Rousseau (2002). "Moore's Law and learning by doing," Review of Economic Dynamics 5, 346-375.

C9. Alternative calibration

In the main text, I set the capital share to $\beta = 0.4$. As a result, the elasticity of output with respect to the aggregate capital stock is $1 - \beta$, which generates a strong external effect. I now consider an alternative calibration where I set $\beta = 0.75$. The rest of the calibration is as in the main text. As shown in Table 3, the bargaining power of brokers is now equal to 99%.

Parameter	Explanation/Target	Value
ρ	rate of time preference	0.01
θ	intertemporal elasticity of substitution	2
β	capital share	0.75
(A_L, A_H)	Dispersion of firm productivities and growth rate	$(0.047, 0.077)$
(π_H, π_L)	Representation of US plant-level productivity as AR(1)	(0.5,0.5)
λ	Persistence of US plant-level productivity	0.5
α	Time to sell machinery equipment	2
η	Capital reallocation	0.99

Table 3. Calibrated parameter values

Figure 17 plots the real interest rate as a function of α and η. The results are qualitatively similar to the ones obtained in the main text. The real interest can vary in a non-monotonic fashion with frictions and it can be larger than its frictionless value.

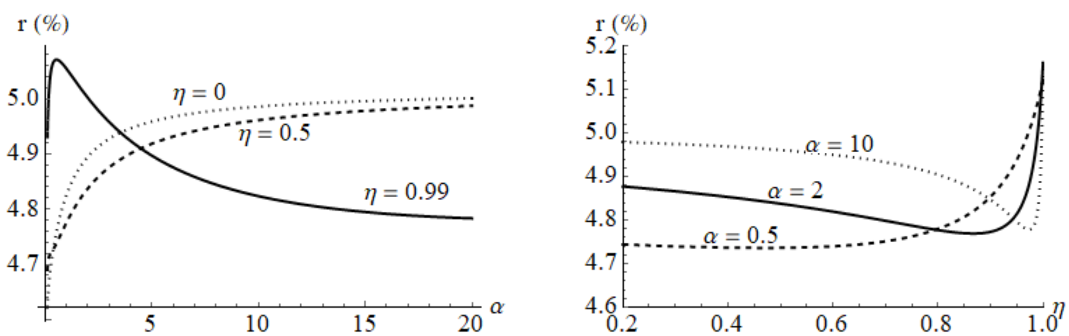

Figure 17: Real interest and market frictions: An alternative calibration

Figure 18 plots the measure, MIS_Y, of static misallocation as a function of market frictions. For a given capital stock, the loss of output when the capital market is illiquid, i.e., α is low, is sizeable. It approaches 8% as α tends to 0. Similarly, MIS_Y increases with brokers' bargaining power and reaches about 4.5% as η tends to one.

Finally, Figure 19 plots the welfare gain, Δ, from removing all market frictions. In contrast to the results in the main text, Δ is always positive. Even though a low α or a high η can stimulate growth, the negative effect on the allocation of capital across firms dominates and welfare falls. The gain from removing frictions can be large. For instance, if α is close to 0 and η close to 1, the welfare gain is greater than 10% of consumption.

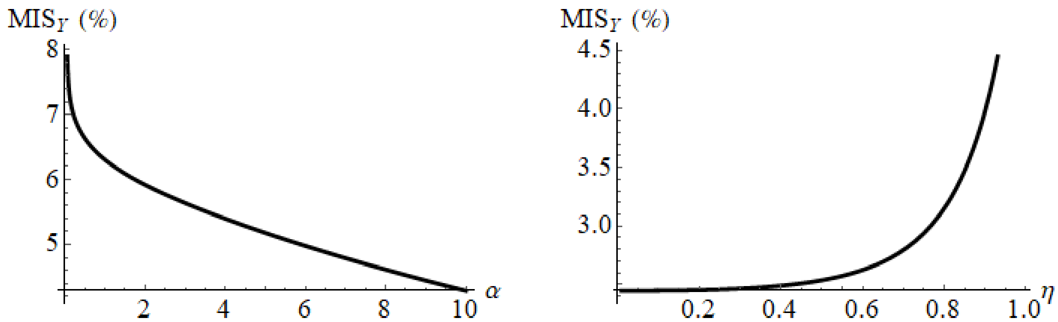

Figure 18: Static misallocation and market frictions: An alternative calibration

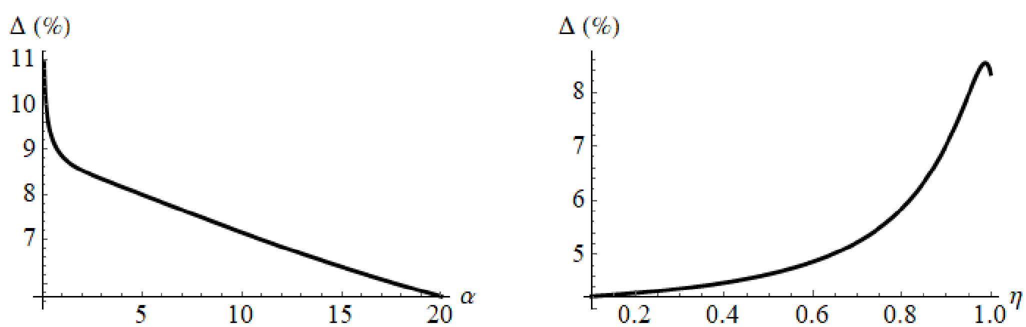

Figure 19: Welfare and market frictions: An alternative calibration

C10. Growth, money, and capital reallocation

In the main text, I introduced a notion of illiquidity in the market for capital goods by assuming that it takes time to trade and reallocate capital goods across firms. An alternative notion of liquidity is as an attribute to an asset that can serve as means of payment, such as money. So far, we have assumed that firms faced no liquidity constraint when financing their acquisition of new capital. They can borrow against their future income and promise to repay their debts. We now relax this assumption and assume that in order to purchase capital, the firm must hold liquid assets, e.g., money or government bonds. It will allow us to revisit a classical question in macroeconomics since Tobin (1965), Sidrauski (1967), Brock (1974), and Stockman (1981), the relationship between money (or liquidity) and economic growth.[35]

Modified environment

I make the following assumptions in order to generate a role for liquidity. Stocks, private debt, or claims on physical capital are not accepted as means of payment in the capital market. The only accepted means of payments are government-issued liabilities, such as money or bonds. Firms can rent liquid assets from financial institutions at the price s in a competitive market opened continuously through time. The renting price of liquidity, $s \geq 0$, is the interest-rate spread between illiquid financial assets (i.e., stocks held by firms) and liquid assets, which represents the cost incurred by financial institutions to hold liquid assets in order to rent them. For instance, if the liquid asset takes the form of money and the inflation rate is $\pi > -r$, then $s = r + \pi$. If the liquid asset takes the form of real government bonds that pays a real interest rate r^g, then $s = r - r^g$. Firms can adjust their holdings of liquid assets in-between meetings with brokers, but liquidity cannot be adjusted instantly at the time of the meetings. Hence, liquidity must be held prior to the meetings.

I generalize the description of firms' idiosyncratic productivity shocks as follows. As before, the most productive firms have a productivity equal to A_H. Each firm receives an opportunity to draw a new productivity at Poisson arrival rate λ from the distribution $G(A) = \pi_H G_H(A) + \pi_L G_L(A)$. The distribution $G_H(A)$ is a step function that assigns mass one to A_H. The distribution $G_L(A)$ is continuous with support $(0, A_H)$. These assumptions imply that the distribution of firms across productivities has a mass point at A_H and is continuous otherwise. We represent the distribution of values for A in Figure 20.

[35] Ireland (1994) revits the relationship between money and growth in the context of an AK model where the payment system is endogenous.

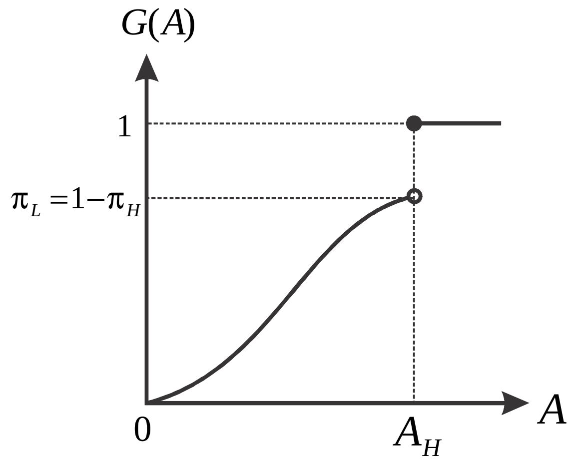

Figure 20: Distribution of productivity shocks.

The bargaining problem

In order to simplify the bargaining problem between the firm and the intermediary, we make the following assumption. While the acquisition of new capital, $k' - k$, must be financed with liquid assets, the payment of the intermediation fees to the brokers can be financed with debt, i.e., the firm can commit to pay ϕ in the future at an interest rate equal to r.[36]

Let us start with the bargaining problem between a firm with capital, k, real money balances, m, and productivity, A, and a broker. The terms of trade solve:

$$(k', \phi) = \arg \max_{k' \in [0, k+m]} \left[\upsilon(k', A) - (k' - k) - \phi - \upsilon(k, A) \right]^{1-\eta} [\phi]^{\eta}. \tag{148}$$

The novelty is that the acquisition of new capital, $k' - k$, must be financed with liquid assets, $k' - k \leq m$. The value function of the firm does not depend on m which is a control variable that can be adjusted instantly at any point in time. By assumption, the payment of ϕ is not subject to a liquidity constraint and can be financed with debt. The outcome of the negotiation is

$$k' \in \arg \max_{k' \in [0, k+m]} \left[\upsilon(k', A) - (k' - k) - \upsilon(k, A) \right] \tag{149}$$

$$\phi = \eta \max_{k' \in [0, k+m]} \left[\upsilon(k', A) - (k' - k) - \upsilon(k, A) \right]. \tag{150}$$

From (149) the choice of capital is the one the firm would make if it had direct access to the capital market. As before, from (150), the intermediation fee represents a fraction η of the gains from trade. It is as if the firm evolves in an economy where brokers have no bargaining power but the contact rate with the market is $\alpha(1 - \eta)$.

[36] See Lagos and Zhang (2020) for a similar assumption.

Firm valuations and Tobin's q

The lifetime expected profits of firms are determined by the following HJB equation:

$$\begin{aligned} rv(k,A) &= \max_{m\geq 0}\left\{-sm + Ak + \lambda\int_{[0,A_H]}\left[v(k,x)-v(k,A)\right]dG(x)\right.\\ &\left. + \alpha(1-\eta)\max_{k'\in[0,k+m]}\left[v(k',A)-(k'-k)-v(k,A)\right]\right\}, \end{aligned} \tag{151}$$

where m denote the choice of liquid assets. The first term on the right side is the cost of renting liquid assets, sm. The acquisition of new capital, $k'-k$, is bounded above by the firm's holdings of liquid asset, i.e., $k'-k\leq m$. The second term corresponds to the arrival of productivity shocks drawn from the distribution G. The last term describes the arrival of a meeting with a broker.

I guess that $v(k,A)=q_A k$. The value functions in (151) can be rewritten as:

$$rq_A k = Ak + \lambda\int_{[0,A_H]}(q_x - q_A)\,kdG(x) + \Omega(k,A), \tag{152}$$

where the last term represents the problem of the firm,

$$\Omega(k,A)\equiv\max_{0\leq k'\leq k+m}\left\{-sm+\alpha(1-\eta)\,(q_A-1)\,(k'-k)\right\}. \tag{153}$$

The firm chooses its holdings of liquid assets and its future capital in order to maximize the value of its investment net of the renting cost of liquidity. If $q_A>1$, then $k'=k+m$ and there is a positive and bounded solution for m if $-s+\alpha(1-\eta)\,(q_A-1)=0$. Assuming q_A is increasing in A, in any monetary equilibrium this equality must hold for the highest productivity, $A=A_H$. If we denote q_H the value of q_A evaluated at $A=A_H$,

$$q_H = 1+\frac{s}{\alpha(1-\eta)}. \tag{154}$$

The Tobin's q of the most productive firms is greater than one in order to compensate for the holding cost of liquid assets that is necessary to finance the acquisition of capital. Note the magnitude of the effect of s on q_H, $\partial q_H/\partial s=\left[\alpha(1-\eta)\right]^{-1}$ depends on the search and bargaining frictions in the market. The effect is stronger if the market is more illiquid, α is low, and if brokers' bargaining power is high, η is close to one. Assuming (154) holds,

$$\Omega(k,A)=\alpha(1-\eta)\,(1-q_A)\,k\mathbb{I}_{\{q_A\leq 1\}}.$$

So, $\Omega(k,A)$ is proportional to k and from (152) the conjecture $v(k,A)=q_A k$ is verified.

We divide both sides of (152) to obtain:

$$rq_A = A+\lambda\pi_H\,(q_H-q_A)+\lambda\pi_L\int(q_x-q_A)\,dG_L(x)+\alpha(1-\eta)\,(1-q_A)\,\mathbb{I}_{\{q_A\leq 1\}}. \tag{155}$$

The last term on the right side of (155) captures the gain when firms liquidate their capital. The following Lemma gives a closed-form solution for q_A.

Lemma 12 ***(Tobin's*** q ***and liquidity.)*** *The Tobin's* q *of a firm with productivity* A *is given by*

$$q_A = 1 + \frac{r+\lambda}{r+\lambda+\alpha(1-\eta)\mathbb{I}_{\{A<B\}}}\left[\frac{s}{\alpha(1-\eta)} - \frac{A_H - A}{r+\lambda}\right], \tag{156}$$

where B*, the productivity level such that* $q_B = 1$*, solves*

$$\frac{s}{\alpha(1-\eta)} = \frac{A_H - B}{r+\lambda}. \tag{157}$$

Proof of Lemma 12. Let B the value of productivity such that $q_B = 1$. From (155),

$$\frac{\partial q_A}{\partial A} = \frac{1}{r+\lambda+\alpha(1-\eta)\mathbb{I}_{\{A\leq B\}}}.$$

So, the value function is piecewise linear with a kink at $A = B$. By integrating $\partial q_A/\partial A$ over the intervals (A, A_H) and (A, B), we obtain:

$$q_A = \begin{cases} q_H - \frac{A_H - A}{r+\lambda} \\ q_B - \frac{B-A}{r+\lambda+\alpha(1-\eta)} \end{cases} \forall A \in \begin{cases} (B, A_H) \\ (0, B). \end{cases} \tag{158}$$

Using that $q_B = 1$ and $q_H = 1 + s/\alpha(1-\eta)$, it follows from (158) that B is determined by (157). From (158) and (157), the expression for the Tobin's q can be written more compactly as (156). ■

For all $A < B$, firms liquidate their capital whenever they access a broker whereas for all $A > B$, they hoard their capital. For given r, the productivity threshold above which firms hoard their capital, B, increases with s. From (156), the passthrough from the liquidity spread, s, to the Tobin's q increases with the productivity level. Note that r is endogenous and will also depend on s.

Equilibrium real interest rate

We close the description of the equilibrium with the determination of the real interest rate. In the case of $A = A_H$, the HJB equation (155) simplifies to

$$rq_H = A_H + \lambda\pi_L \int (q_x - q_H)\, dG_L(x). \tag{159}$$

By plugging q_H given by (154) into (159), the real interest rate is determined by

$$r\left(1 + \frac{s}{\alpha(1-\eta)}\right) = A_H - \lambda\pi_L \int_0^B \left(\frac{A_H - A + s}{r+\lambda+\alpha(1-\eta)}\right) dG_L(A) \tag{160}$$

$$-\lambda\pi_L \int_B^{A_H} \frac{A_H - A}{r+\lambda} dG_L(A).$$

An equilibrium is a pair, (r, B), solution to (157)-(160). When liquidity is costless, $s = 0$, $B = A_H$ and r solves

$$r = A_H - \frac{\lambda \pi_L}{r + \lambda + \alpha(1-\eta)} \int_0^{A_H} G_L(A) dA.$$

It corresponds to (24) when G_L is a step function.

Proposition 16 ***(The cost of liquidity and economic growth.)*** *As s increases, the rate of return of capital and the growth rate of the economy decreases.*

In order to illustrate this result, we return to the case of two productivity levels by assuming that G_L is a Dirac measure assigning mass one to $A_L \in (0, A_H)$. We obtain the following Corollary.

Corollary 3 *Suppose $G_L(A) = \mathbb{I}_{\{A \geq A_L\}}$. There is a unique balanced-growth-path equilibrium and it is such that as s increases, r and g decrease while q_L and q_H increase. Moreover, there is a pair, $(s_0, r_0) \in \mathbb{R}_+^2 > 0$, solution to*

$$r_0 (r_0 + \lambda) = (r_0 + \lambda \pi_L) A_L + \lambda \pi_H A_H \tag{161}$$

$$s_0 = \frac{\alpha(1-\eta)(A_H - A_L)}{r_0 + \lambda}, \tag{162}$$

such that the following is true:

1. *For all $s < s_0$, the equilibrium features $q_L < 1$, i.e., low-productivity firms liquidate their capital when given the opportunity. The share of the aggregate capital stock held by low-productivity firms is*
$$\hat{k}_L = \frac{\lambda \pi_L}{g + \alpha + \lambda}.$$
2. *For all $s > s_0$, the equilibrium features $q_L > 1$, i.e., low-productivity firms hold onto their capital. The share of the aggregate capital stock held by low-productivity firms is*
$$\hat{k}_L = \frac{\lambda \pi_L}{g + \lambda}.$$

Proof of Corollary 3. From (160), the real interest rate, r, is the solution to:

$$r \left[1 + \frac{s}{\alpha(1-\eta)} \right] = A_H - \frac{\lambda \pi_L (A_H - A_L + s)}{r + \lambda + \alpha(1-\eta)} \quad \text{if } r < \hat{r} \tag{163}$$

$$= A_H - \frac{\lambda \pi_L (A_H - A_L)}{r + \lambda} \quad \text{if } r > \hat{r}, \tag{164}$$

where, from (157),

$$\hat{r} \equiv \frac{\alpha(1-\eta)\left(A_H - A_L\right)}{s} - \lambda. \tag{165}$$

First, we establish that there is a unique s, denoted s_0, such that $r = \hat{r}$. We denote $r_0 = \hat{r}(s_0)$. The pair, (s_0, r_0), solves (164) and (165), i.e.,

$$\begin{aligned} r_0\left[1 + \frac{s_0}{\alpha(1-\eta)}\right] &= A_H - \frac{\lambda\pi_L(A_H - A_L)}{r_0 + \lambda} \\ r_0 &= \frac{\alpha(1-\eta)\left(A_H - A_L\right)}{s_0} - \lambda. \end{aligned}$$

They can be rearranged to correspond to (161)-(162).

Next, we establish that there is a unique r solution to (163)-(164). Denote r_1 the solution to (163) and r_2 the solution to (164). Both solutions are unique and belong to the interval $(0, A_H)$. Moreover, both solutions are decreasing functions of s. For all $r < \hat{r}$,

$$\frac{\lambda\pi_L\left(A_H - A_L + s\right)}{r + \lambda + \alpha(1-\eta)} < \frac{\lambda\pi_L(A_H - A_L)}{r + \lambda}.$$

It follows that if $r_1 < \hat{r}$, then $r_2 < \hat{r}$ and hence r_2 cannot be a solution to (163)-(164). Conversely, if $r_2 > \hat{r}$, then $r_1 > \hat{r}$ and hence r_1 cannot be the solution (163)-(164).

If $s < s_0$, the dynamics of the capital held by high-productivity firms is given by (16). The Tobin's q for low-productivity firms solves

$$rq_L = A_L + \lambda\pi_H\left(q_H - q_L\right) + \alpha(1-\eta)\left(1 - q_L\right).$$

Solving for q_L and substituting q_H by its expression:

$$q_L = \frac{A_L + \lambda\pi_H q_H + \alpha(1-\eta)}{r + \lambda\pi_H + \alpha(1-\eta)}.$$

As s increases, q_H increases and r decreases. Hence, q_L increases.

If $s > s_0$, the capital stocks in high- and low-productivity firms evolve according to:

$$\begin{aligned} \dot{K}_H &= \lambda\pi_H K_L - \lambda\pi_L K_H + \dot{K} \qquad (166) \\ \dot{K}_L &= \lambda\pi_L K_H - \lambda\pi_H K_L. \qquad (167) \end{aligned}$$

According to (166), the stock of capital held by high productivity firms increases with the flow flow of low-productivity firms that receive a high productivity shock, it decreases with the flow of high-productivity firms that receive a low productivity shock, and it increases with the flow of new capital goods entering the market. Using that $\dot{K}_H/K_H = \dot{K}_L/K_L = \dot{K}/K$, it follows that the capital share are equal to:

$$\begin{aligned} \hat{k}_H &\equiv \frac{K_H}{K} = \frac{\lambda\pi_H + g}{\lambda + g} \\ \hat{k}_L &\equiv \frac{K_L}{K} = \frac{\lambda\pi_L}{g + \lambda}. \end{aligned}$$

The Tobin's q for low-productivity firms solves

$$rq_L = A_L + \lambda\pi_H (q_H - q_L).$$

Solving for q_L and substituting q_H by its expression:

$$q_L = \frac{A_L + \lambda\pi_H q_H}{r + \lambda\pi_H}.$$

As s increases, q_H increases and r decreases. Hence, q_L increases. ■

If the cost of holding liquidity, s, is smaller than some threshold, s_0, then q_L is less than one so that low-productivity firms liquidate their capital when they access the capital market. As s increases, the rate of return of capital decreases and hence the growth rate of the economy decreases. Moreover, the Tobin's qs, q_H and q_L, increase. The distribution of capital across productivity types is the same as the one in the economy without liquidity constraints. The cost of liquidity only affects the misallocation of capital, $\hat{k}_L$, through g, i.e., a larger share of the capital is misallocated as s increases because the rate at which new capital is formed slows down.

If s is larger than s_0, then the equilibrium is such that $q_L > 1$, in which case low-productivity firms choose not to liquidate their capital. The fact that low-productivity firms hoard their capital raises the misallocation of capital relative to an economy with a low cost of liquidity.

From (162) the threshold s_0 decreases with η. So, as brokers' bargaining power increases, the economy is more likely to be in the economy where low-productivity firms hoard capital instead of selling it. As a result, in the presence of liquidity constraints, η affects both the rate at which capital is accumulated and its distribution across firms.

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